LINFO1115
Reasoning About a Highly Connected World

Course slides

These slides are based on the book "Networks, Crowds, and Markets" by David Easley and Jon Kleinberg (2010, Cambridge University Press). Lecture 10 is by Sarunas Girdzijauskas (Royal Institute of Technology, Stockholm, Sweden).

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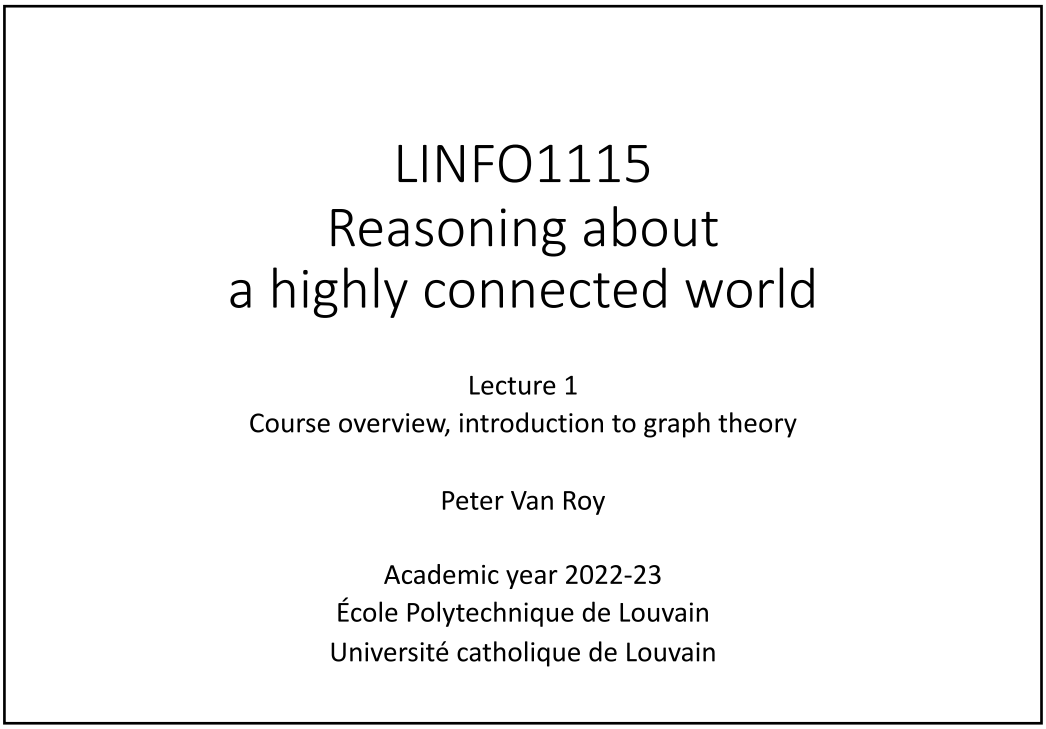
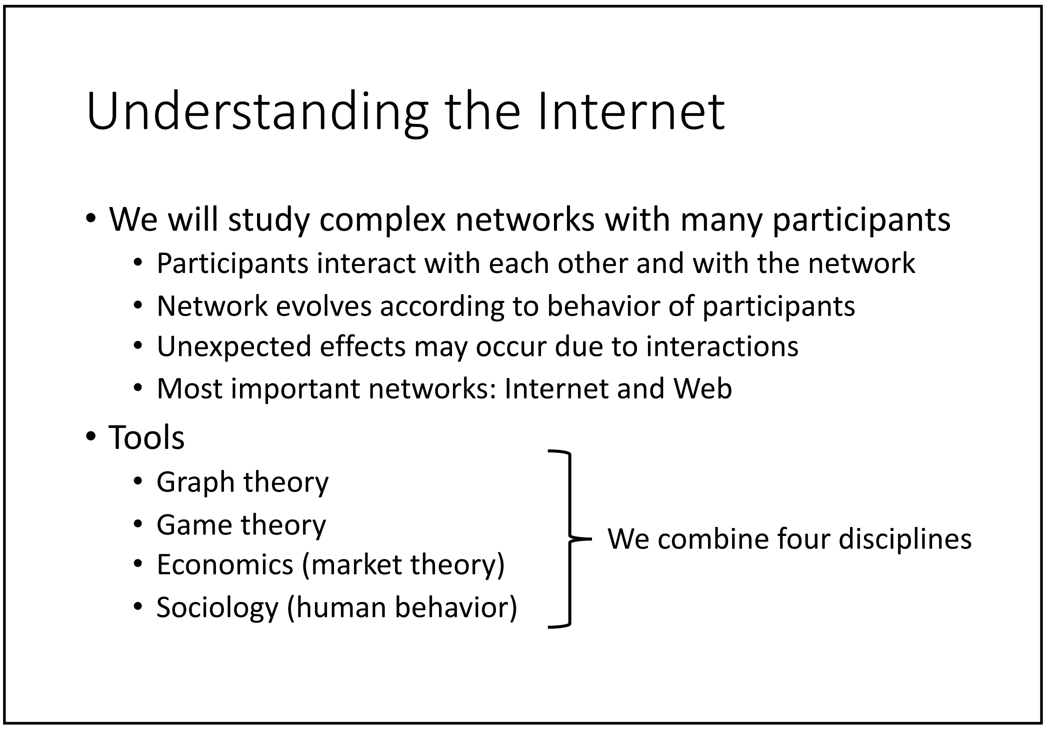


LINFO1115
Reasoning about a highly connected world

Lecture 1
Course overview, introduction to graph theory

Peter Van Roy

Academic year 2022-23
École Polytechnique de Louvain
Université catholique de Louvain



Understanding the Internet

- We will study complex networks with many participants
  - Participants interact with each other and with the network
  - Network evolves according to behavior of participants
  - Unexpected effects may occur due to interactions
  - Most important networks: Internet and Web
- Tools
  - Graph theory
  - Game theory
  - Economics (market theory)
  - Sociology (human behavior)

We combine four disciplines

Approach used in the course

- How do we study complex networks?
  - They are too complicated to analyze directly!
- Three-step approach used throughout the course
  1. Define simple models with just the right amount of mathematics
  2. Study the models and analyze how they work
  3. Get intuition on how the complex network works
- For example, social networks
  - A graph where nodes are humans and edges are friend/enemy links
  - Study how these graphs evolve over time
  - Closure: friends of friends can become friends
  - Structural balance: friend/enemy networks evolve toward balance
- We will define many simple models
  - Around fifteen models to help understand many aspects of the Internet

Course organization

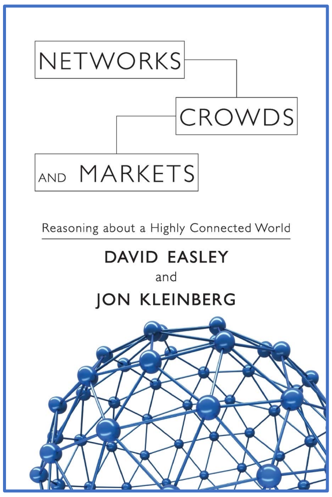
All course information on Moodle!

- Teaching
  - Weekly lecture: Wednesday 14h-16h (BA91)
  - Weekly lab: Monday 16h15-18h15 (starts S2) (BA91)
  - Optional midterm: worth 5 points (dispensatory)
  - Mandatory project: groups of 2, worth 5 points on final grade
  - Final exam: worth 15 points (first 5 points: max of midterm grade)
  - Bonus point: 1 extra point if you attend all lab sessions
- Team
  - Peter Van Roy (professor) peter.vanroy@uclouvain.be
  - Lucile Dierckx (teaching assistant) lucile.dierckx@uclouvain.be
  - Tom Lai (tutor) tom.lai@student.uclouvain.be
  - Nathan Nepper (tutor) nathan.nepper@student.uclouvain.be

Course topics

- Graph theory (weeks 1-3)
    - Paths and connectivity
    - Structure of social networks
    - Strong and weak links
    - Affiliation and closure
    - Positive and negative links
- Game theory (weeks 4-5)
    - Games, best responses, and dominant strategies
    - Equilibria and optimality
    - Mixed strategies
    - Traffic and Braess's paradox
    - Auctions
- Markets (weeks 6-8)
    - Matching markets
    - Market clearing
    - Intermediaries
    - Bargaining
- Internet + Web (weeks 9-13)
    - Web bow-tie structure
    - Web search and PageRank
    - Sponsored search markets
    - Information cascades
    - Network effects
    - Power laws
    - Structural cascades
    - Small-world phenomenon

Course syllabus



- This book is the mandatory syllabus for the course
    - The slides have much information but are very concise! The book has lots of good explanations.
- 10 copies are available to borrow at the BST
- We recommend that you buy your own copy
    - It is a really good book!

Book chapters

- Graph theory (weeks 1-3, **ch 2-5**)
  - Paths and connectivity
  - Structure of social networks
  - Strong and weak links
  - Affiliation and closure
  - Positive and negative links
- Game theory (weeks 4-5, **ch 6, 8, 9**)
  - Games, best responses, and dominant strategies
  - Equilibria and optimality
  - Mixed strategies
  - Traffic and Braess's paradox
  - Auctions
- Markets (weeks 6-8, **ch 10-12**)
  - Matching markets
  - Market clearing
  - Intermediaries
  - Bargaining
- Internet + Web (weeks 9-13, **ch 13-20**)
  - Web bow-tie structure
  - Web search and PageRank
  - Sponsored search markets
  - Information cascades
  - Network effects
  - Power laws
  - Structural cascades
  - Small-world phenomenon

Teasers

Graphs, games, markets, Web

Graph theory teaser

Social networks as graphs

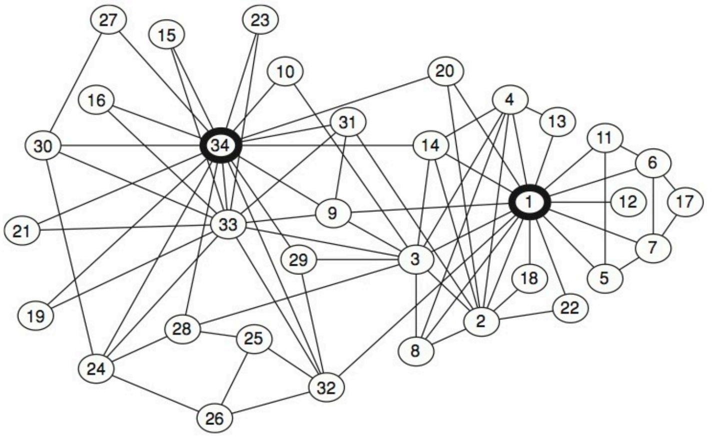
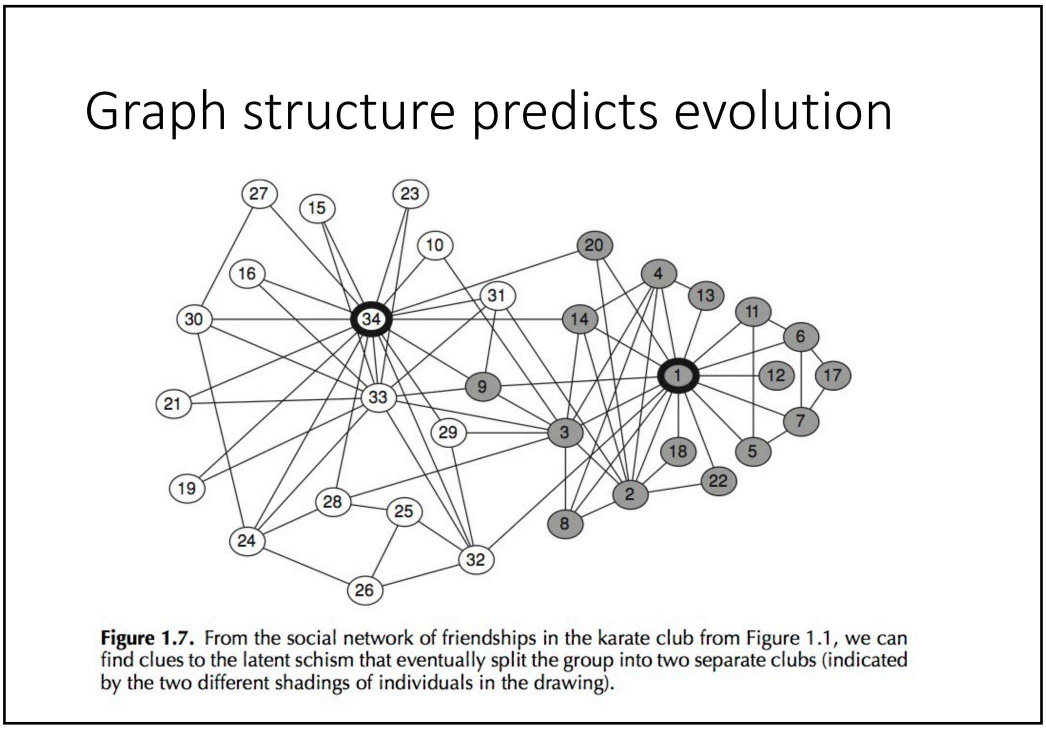
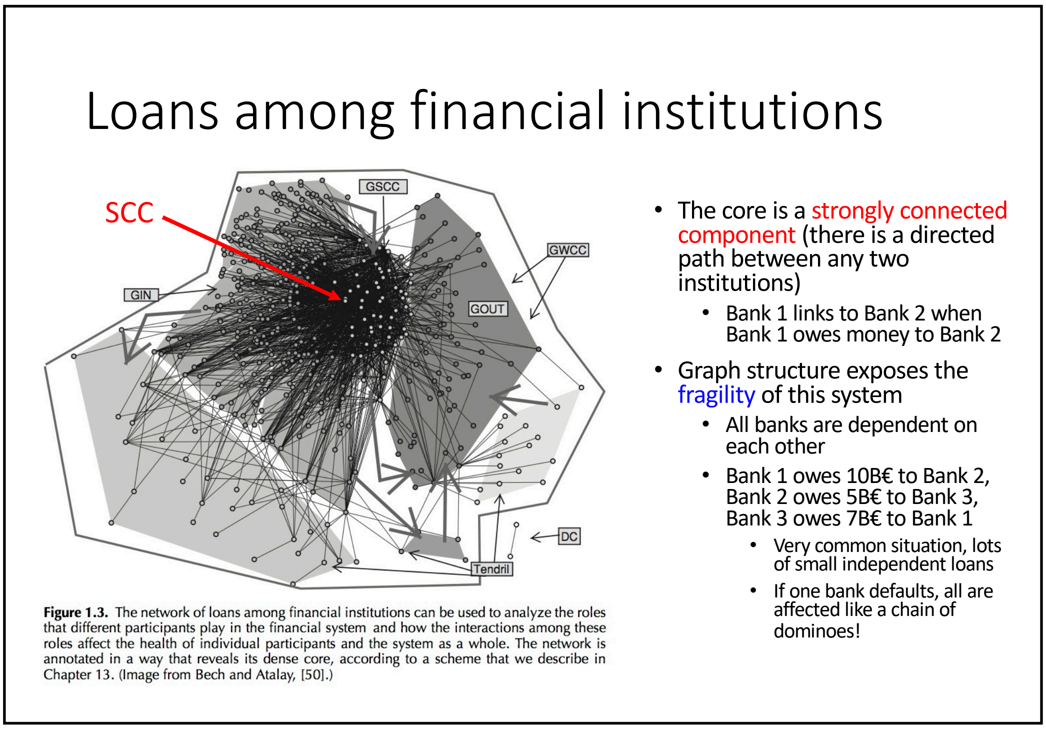


Figure 1.1. The social network of friendships within a 34-person karate club [421]. (Drawing from the *Journal of Anthropological Research*.)



Graph structure predicts evolution

Figure 1.7. From the social network of friendships in the karate club from Figure 1.1, we can find clues to the latent schism that eventually split the group into two separate clubs (indicated by the two different shadings of individuals in the drawing).



Loans among financial institutions

Figure 1.3. The network of loans among financial institutions can be used to analyze the roles that different participants play in the financial system and how the interactions among these roles affect the health of individual participants and the system as a whole. The network is annotated in a way that reveals its dense core, according to a scheme that we describe in Chapter 13. (Image from Bech and Atalay, [50].)

- The core is a strongly connected component (there is a directed path between any two institutions)
  - Bank 1 links to Bank 2 when Bank 1 owes money to Bank 2
- Graph structure exposes the fragility of this system
  - All banks are dependent on each other
  - Bank 1 owes 10B€ to Bank 2, Bank 2 owes 5B€ to Bank 3, Bank 3 owes 7B€ to Bank 1
    - Very common situation, lots of small independent loans
    - If one bank defaults, all are affected like a chain of dominoes!

Game theory teaser

Game theory

- Game theory models situations where players interact with each other and each player's outcome depends on the decisions made by the others
- It's not just about playing games for fun!
  - How to price a new product when other firms have similar products (game = companies selling products to consumers)
  - How to bid in an auction (game = buyers competing for one item)
  - How to choose a route on the road or on the Internet (game = cars sharing roads or packets on a network)
  - Whether to be passive or aggressive in international relations (game = power politics of countries)
- Games can involve many players in complex situations
  - But let's start by looking at a simple game…

A simple game

I prepare the presentation.
Partner studies for exam.
I get (80 + 92)/2 = 86%
Partner gets (92 + 92)/2 = 92%

		Your partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

- You and your partner are students
  - Tomorrow each of you has a joint presentation and an individual exam
  - Do you study for the exam or prepare the presentation?
- Exam: if you study, you get 92%, if not you get 80%
- Presentation: if both prepare, you both get 100%
  - If only one prepares, you both get 92%
  - If nobody prepares, you both get 84%
- Total grade is average of exam and presentation
- What do you do? Study for the exam or prepare the presentation?

A simple game

I prepare the presentation.
Partner studies for exam.
I get (80 + 92)/2 = 86%
Partner gets (92 + 92)/2 = 92%

Each student decides independently. What should I do?

If partner prepares then I should study (92 versus 90)!
If partner studies then I should study (88 versus 86)!
So I study, right?

		Your partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

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		Your partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

If partner prepares then I should study (92 versus 90)!
If partner studies then I should study (88 versus 86)!
So I study, right?

But if both prepare then they get more!

- You and your partner are students
  - Tomorrow each of you has a joint presentation and an individual exam
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- Exam: if you study, you get 92%, if not you get 80%
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- Total grade is average of exam and presentation
- What do you do? Study for the exam or prepare the presentation?

Markets teaser

Markets on networks

- Markets are a main example of interactions on networks
  - Trade with buyers and sellers
  - Offering services and using services
  - Offering information and searching information
- What kinds of networks?
  - Web and Internet
  - Geographical networks and trade routes
- How do markets work?
  - Market clearing: supply and demand make an equilibrium
  - Intermediaries: traders and stock markets
  - Bargaining and power: depends on position in the network!
  - Network effects: better when many users (more apps, etc.)

Stock markets and order books

ASK: Will sell for $5.00
(ASK = lowest to sell)

BID: Will buy at $4.00
(BID = highest to buy)

Trader has bought
shares at $5.00
(market order)

- Stock markets are one kind of market with intermediaries
  - Sellers – Traders – Buyers
- For each stock that is traded, the market creates an order book
  - An order book is a list of orders that buyers and sellers have submitted
  - Lowest offer to sell is called the ask
  - Highest offer to buy is called the bid
- Order book is managed by a specialist (“trader”)
  - Traders make offers to buy and sell, for example at market prices (market order)
  - If you want to buy or sell stock, contact a trader (talk to your bank)!

Web teaser

Example: bow-tie structure of Web

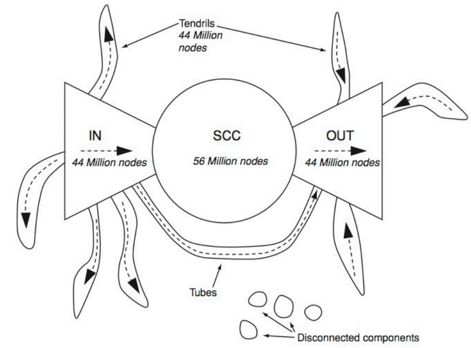


Figure 13.7. A schematic picture of the bow-tie structure of the Web (image from Broder et al., [80]). Although the numbers are now outdated, the structure has persisted.

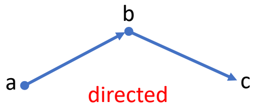
- The Web has grown organically for thirty years without any central organization
- Can we say anything about its global structure?

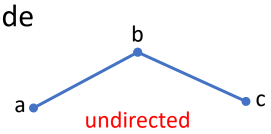
Introduction to graph theory

Chapter 2

Basics of graphs

- Definition of a graph
  - A graph G = (N,E) with N a set of nodes and E a set of edges ("arêtes"), where an edge is a pair of two nodes
  - Example: N={a,b,c}, E={{a,b},{b,c}}
- Directed graph ("orienté")
  - Each edge has a direction {(a,b),(b,c)}
  - An edge is a tuple with first and second node
- Undirected graph ("non-orienté")
  - Edges have no direction {{a,b},{b,c}}
  - An edge is a set of two nodes
  - Most of the graphs in this course are undirected, but not all!





Example: ARPANET

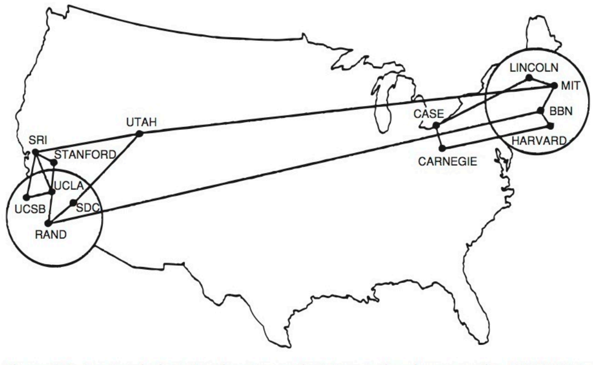


Figure 2.2. A network depicting the sites on the Internet, then known as the ARPANET, in December 1970. (Image from F. Heart, A. McKenzie, J. McQuillian, and D. Walden, [214]; available online at http://som.csudh.edu/cis/lpress/history/arpamaps/.)

- A graph can model a network
- Communication network: ARPANET (early Internet)
  - Node = computer or institution
- Social network
  - Node = person
- Information network
  - Node = information source (Web page or document)

ARPANET without the geography

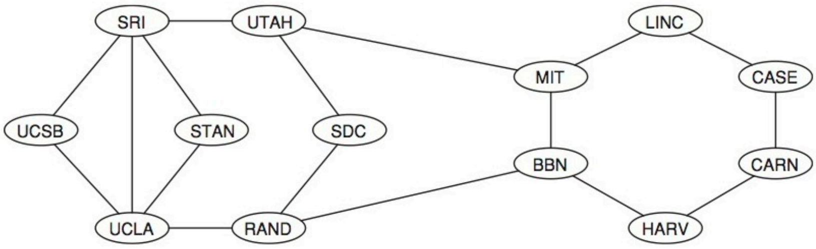


Figure 2.3. An alternate drawing of the thirteen-node Internet graph from December 1970.

- A graph is an abstract structure
  - In this course it often represents ("models") a property of a real-world structure

Paths and connectivity (1)

- Definition of a path
  - A path in a graph is a sequence of nodes such that each successive pair in the sequence is an edge of the graph
- Definition of a simple path
  - A simple path is a path where each node occurs at most once
- Definition of a cycle
  - A cycle is a path with ≥3 edges such that the first and last nodes are the same and otherwise all nodes are distinct
    - Why at least 3 edges? We are not interested in trivial cycles!
    - This is almost a simple path. What is the difference?
  - Every edge in the ARPANET graph is part of a cycle, which gives redundancy: more than one path connecting each pair of nodes

Paths and connectivity (2)

- Definition of a connected graph
  - A graph is connected (“connexe”) if there exists a path between every pair of nodes
- Definition of a component of a graph
  - A component is a subset of nodes that satisfies two properties:
    - The subset is connected
    - The subset is maximal: there is no superset that is connected
  - Both conditions are important!
- Analyzing a graph
  - A graph can be divided into its components
  - For each component, we can study its internal structure. For example, removing a node (with its edges) may split it into two components.

Giant components

- Consider a social network where the nodes are all people on Earth and there is an edge between two people if they are friends
  - Is this graph connected? Probably not! (Isolated tropical island)
  - But still, many people have friends far away
- Large social networks often have a giant component, which is a component that contains a significant fraction of the graph's nodes. This is a deliberately informal concept (not precise)!
  - If there exists a giant component, then there is almost always only one. Reasoning: if there were two, then adding just one edge between them would cause them to fuse, "collapsing" into one component.
- Real-world examples of collapsing giant components are often significant events and even catastrophic
  - When European explorers arrived in the Americas, this caused such a collapse, with dramatic effects on technology and disease (epidemics)

Example of a giant component

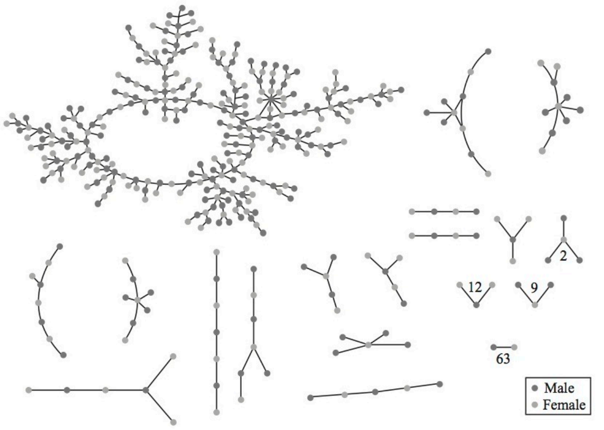


Figure 2.7. A network in which the nodes are students in a large American high school, and an edge joins any two who had a romantic relationship at some point during the 18-month period in which the study was conducted [49]. (Image from The University of Chicago Press.)

- Romantic relationships between high school students
- One giant component, many small components
- Important for transmissible diseases!

Distance in a graph

- Definition of the length of a path
  - Number of edges in the path
- Definition of the distance between two nodes
  - Length of the shortest path between them
- Breadth-first search
  - A systematic way of determining distances is by first finding all nodes at distance 1, then all nodes at distance 2, and so on, increasing by one link each time
  - The approach of traversing the graph in layers that are gradually increasing in distance is called breadth-first search

Breadth-first search

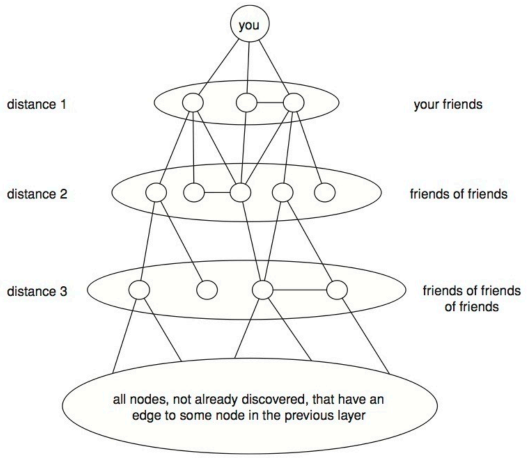


Figure 2.8. Breadth-first search discovers the distances to nodes one "layer" at a time; each layer is built of nodes that have an edge to at least one node in the previous layer.

- First your friends (distance 1)
- Then friends of friends (distance 2)
- Then friends of friends of friends (distance 3)
- Each successive layer adds only the nodes that were not added in a previous layer

Breadth-first search of ARPANET

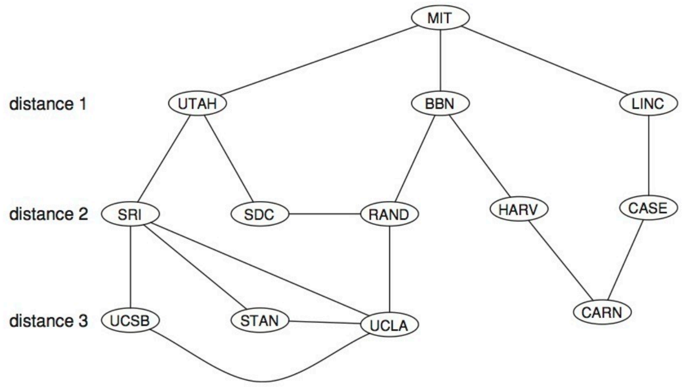


Figure 2.9. The layers arising from a breadth-first of the December 1970 ARPANET, starting at the node MIT.

- We do a breadth-first search of the ARPANET starting from MIT

Small-world phenomenon (1)

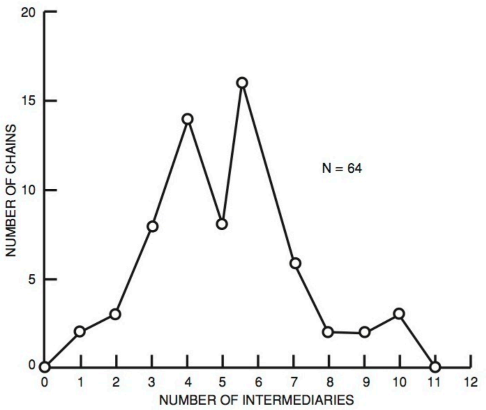


Figure 2.10. A histogram from Travers and Milgram's paper on their small-world experiment [391]. For each possible length (labeled "number of intermediaries" on the *x*-axis), the plot shows the number of successfully completed chains of that length. In total, sixty-four chains reached the target person, with a median chain length of six. (Image from the American Sociological Association.)

- Recall the giant component of all people on earth
  - It turns out that most distances are short: "six degrees of freedom"
- Famous Travers and Milgram experiment: see histogram
  - 300 randomly chosen people forwarded a letter to a target person by giving it to a friend and asking them to continue forwarding it
  - "six short steps" or "six worlds apart"?

Small-world phenomenon (2)

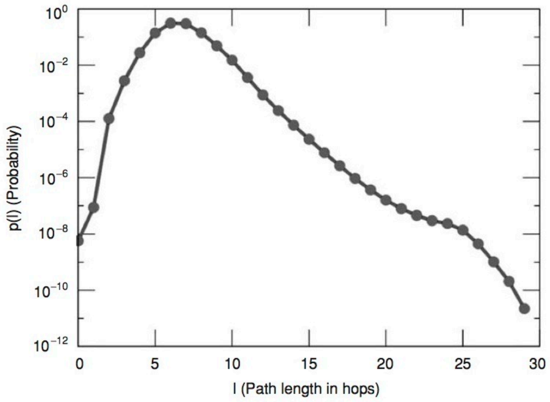


Figure 2.11. The distribution of distances in the graph of all active Microsoft Instant Messenger user accounts, in which an edge joins two users if they communicated at least once during a month-long observation period [273].

- Analysis of 240 million active accounts on Microsoft Instant Messenger (IM)
  - Edge between two users if they had a two-way conversation in a one-month period
- "Erdös number": distance between a researcher and Paul Erdös, where each edge is being coauthor of a paper
  - Most mathematicians have Erdös number of maximum 5
    - Vincent Blondel: 2, Peter Van Roy: 4
- "Bacon number": distance between an actor and Kevin Bacon
  - Average Bacon number for all actors on IMDb is 2.9

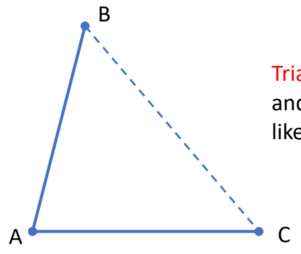
Strong and weak ties (“social networks as graphs”)

Chapter 3

Reasoning about social networks

- Social networks are graphs where the nodes denote human beings and the edges denote a connection between two human beings (such as friendship)
  - Understanding social networks gives insights into society
  - So let us take a closer look at social networks!
- We will combine ideas from two disciplines
  - Graph theory: structural properties of graphs
  - Sociology: behavior of human beings

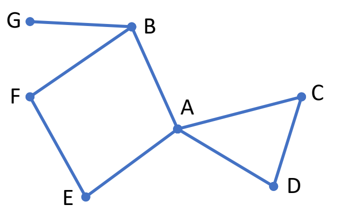
Triadic closure (sociological concept)



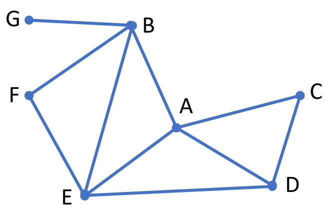
Triadic closure: If A and B are friends, and A and C are friends, then it is more likely that B and C will become friends

- If two people in a social network have a common friend, then the likelihood that they will become friends in the future is increased
- Why? There are several reasons:
  - B and C have more opportunities to meet;
  - B and C both trust A so they have a basis for trusting each other;
  - B and C have common interests with A so they may have common interests with each other;
  - A has an incentive to bring B and C together since if they are not friends, it stresses A.

Clustering coefficient



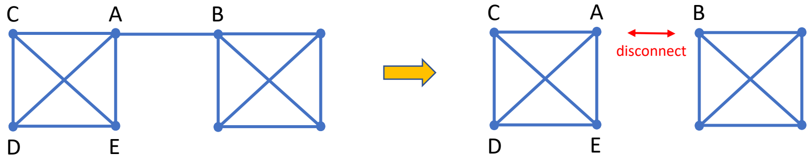
CC(A)=1/6



CC(A)=3/6=1/2

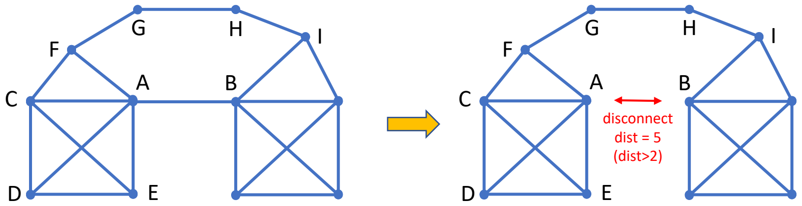
- Definition of the clustering coefficient CC(A) of node A in a graph
  - CC(A) = probability that two randomly selected friends of A are friends with each other
  - CC(A) = $\frac{\text{(the number of pairs of A's friends that are connected)}}{\text{(the maximum number of pairs of A's friends that are connected)}}$
- Triadic closure causes the clustering coefficient to increase

Bridges (graph concept)



- Definition of a bridge
  - The edge joining A and B is a bridge if removing this edge causes A and B to be in different components.
- Let's study person A and their friends
  - A forms a tightly knitted group with C, D, E but the connection to B is different, it reaches into a different part of the network
  - B has access to a whole different set of people, and can give information to A that C, D, and E cannot.

Local bridges (graph concept)



- In real social networks, bridges will probably be rare
  - Because there are very many connections in the connect (small world!), there will probably be other paths from A to B
  - We would like to capture the idea of a bridge, but in a real social network
- Definition of a local bridge
  - The edge joining A and B is a local bridge if A and B have no friends in common, so deleting the edge will increase the distance between A and B to a value strictly greater than 2
  - Local bridges play roughly the same role as bridges, but in a less extreme way: they connect a node to another node that would otherwise be far away

Granovetter problem: friends vs. acquaintances

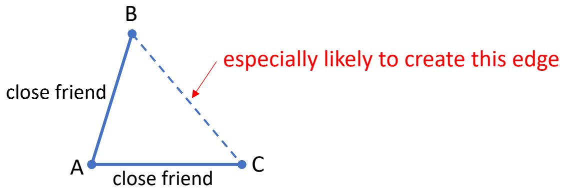
The Granovetter problem

- Consider the following problem:
  - If you interview people who have recently changed jobs, you observe that they usually found their new job through acquaintances, not through close friends
  - Why is that? Close friends have more motivation to help you, so why did the person find the job from an acquaintance? It seems counterintuitive.
- This was studied by Mark Granovetter in the 1960s
  - Let us follow Granovetter's reasoning, using a combination of graph properties and ideas from sociology
  - We will use the idea of edge strength: strong edge (close friend) versus weak edge (acquaintance)

Granovetter's key insight

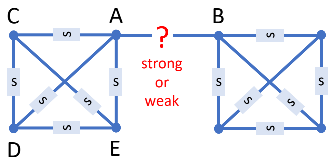
- First observation: (graph theoretical)
  - If A wants to find a new job, it will be through someone connected via a local bridge
    - Because this person has access to new information
- Second observation: (sociological)
  - If someone is connected via a local bridge, it will be an acquaintance rather than a close friend
    - Because otherwise there would be triadic closure
- Conclusion:
  - If A wants to find a new job, it will be through an acquaintance rather than a close friend

Strong triadic closure



- Let’s distinguish between strong edges (close friends) and weak edges (distant acquaintances)
- We can connect this with triadic closure:
  - Triadic closure works better with close friends than with distant acquaintances
  - We call this strong triadic closure: if A and B are close friends and A and C are close friends, then it is especially likely that B and C will become connected

Local bridges are acquaintances (conclusion of Granovetter problem)



- A is close friends with C, D, E so those edges are strong
- What about the local bridge from A to B? Is it strong or weak?
  - It has to be weak! Reasoning from contradiction: if it were strong, then strong triadic closure would make links from B to C, D, E, and it would not be a local bridge.
- This makes a connection between a global property of the graph (local bridge) and a purely local property of edges (strong or weak)
  - Assuming strong triadic closure, local bridges must be weak!
  - Therefore A and B are acquaintances (not close friends)

Generalized measures of tie strength and bridges

Generalizing tie strength and local bridges

- Tie strength and local bridges are very useful concepts
  - But they are too strict: they don't always exist cleanly on big realistic networks
- In large, realistic networks, we need to be more flexible
  - Instead of strong/weak links, we use a numerical quantity for the tie strength
  - Instead of local bridges that cleanly separate a network, we define a measure of how local the bridge is ("almost" a local bridge)
- Let's do this and see what it shows on some real networks

Neighborhood overlap

- Neighborhood overlap generalizes local bridges

$$\frac{\text{(number of nodes who are neighbors of both A and B)}}{\text{(number of nodes who are neighbors of at least one of A and B)}}$$

- This number is exactly 0 for a local bridge (no neighbors of both!)
- This number is close to 0 for “almost” local bridges

Neighborhood overlap as function of tie strength

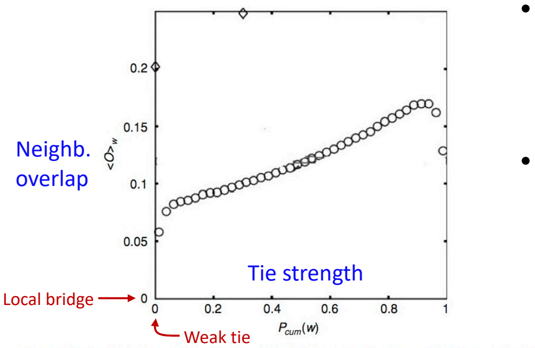


Figure 3.7. A plot of the neighborhood overlap of edges as a function of their percentile in the sorted order of all edges by tie strength. The fact that overlap increases with increasing tie strength is consistent with the theoretical predictions from Section 3.2 [334]. (Image from the National Academy of Sciences, USA.)

- Granovetter says that local bridges are weak ties
  - We can generalize this!
- We can plot neighborhood overlap as a function of tie strength for social networks
  - As we get closer to a local bridge, the tie gets weaker
  - As neighborhood overlap goes to zero, the tie gets weaker
  - Inversely, stronger ties have more neighborhood overlap

Tie strength on Facebook

- On Facebook we can define tie strength depending on how friend links are used:
  - Strongest (mutual communication): messages sent both directions during one month
  - Medium (one-way communication): user sent messages one way, but no reply
  - Weaker (maintained relationship): user follows a friend, but no actual communication taking place
  - Weakest (declared friendship): declared as friend but no following

Facebook user’s neighborhood

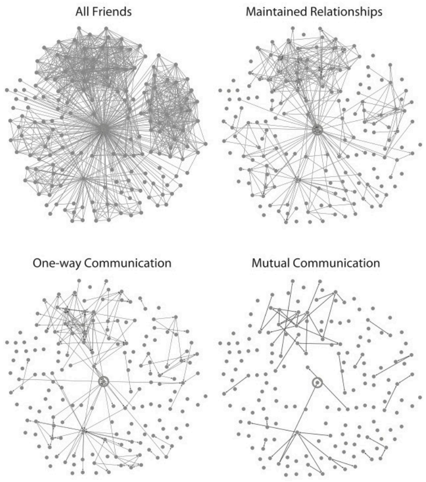
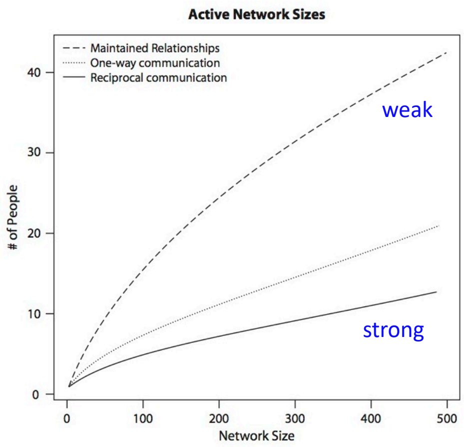


Figure 3.8. Four different views of a Facebook user’s network neighborhood, showing the structure of links corresponding respectively to all declared friendships, maintained relationships, one-way communication, and reciprocal (i.e., mutual) communication, all over a one-month observation period. (Image from [286].)

Facebook neighborhood sizes as a function of total user's network

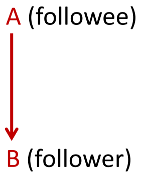


Weak ties: more people, more local bridges

Strong ties: less people, tighter connections

Figure 3.9. The number of links corresponding to maintained relationships, one-way communication, and reciprocal communication as a function of the total neighborhood size for users on Facebook. (Image from [286].)

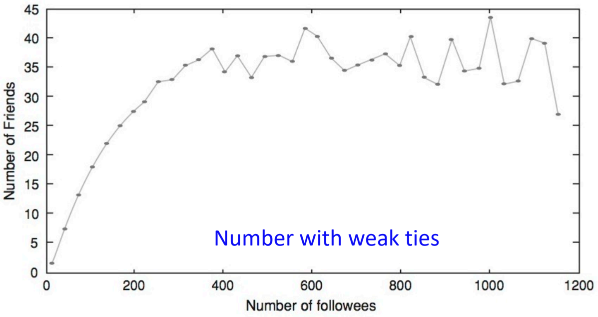
Tie strength on Twitter


A (followee)

B (follower)

- Twitter is a microblogging service where users exchange small messages ("tweets")
  - A user can be a follower of others: receives messages
  - A user can be a followee to others: sends messages
- Tie strength can be defined:
  - Weak if a user is a follower of another
  - Strong if a user sends an individual message to another
- We can measure the total number of links depending on strength

Twitter: few strong links, many weak



Number with strong ties

Number with weak ties

Figure 3.10. The total number of a user's strong ties (defined by multiple directed messages) as a function of the number of followees he or she has on Twitter. (Image from First Monday and [222].)

- Number of friends (mutual messages) as function of number of followees I have (how many I receive from)
- Large number of followees is easy (passive, no work), but number of friends has a maximum (maintaining a friend takes work)

Summary

Summary

- Granovetter’s problem uses two disciplines:
  - Local bridge is a property of a graph (graph theory)
  - Strong triadic closure is a property of friendship (sociology)
  - By combining them, we gain new understanding:
    - People who change jobs will usually find their new job through an acquaintance and not through a close friend
- This is a common theme throughout the course
  - We combine mathematical ideas (graph theory, game theory) with sociological ideas to achieve deeper understanding of Web and Internet
- We first define simple graph concepts (strong/weak, bridge)
  - But in realistic networks we need to generalize the concepts!
  - Strength is a numeric value, local bridge is a numeric value

LINFO1115
Reasoning about
a highly connected world

Lecture 2

Social networks and how they evolve

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Academic year 2022-23

École Polytechnique de Louvain

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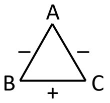
Overview

- Similarity between friends in a social network
  - Homophily: in psychology, this is the principle that we make friends with people similar to us
  - Two mechanisms for homophily: selection (internal) and social influence (external)
- Social-affiliation networks
  - To explain homophily formally, we extend the social network with focus nodes
  - With foci, there are three kinds of closure: triadic closure, focal closure, membership closure
- Empirical studies of social network evolution (closure analysis)
  - Kossinets and Watts studies for triadic closure and for focal closure
  - LiveJournal study and Wikipedia study for membership closure
- Comparing selection and social influence
  - Wikipedia study to compare the effect of selection and social influence
- Friend/enemy networks
  - We extend social networks to have both positive and negative links (friends and enemies)
  - Structural balance: networks evolve to minimize stress (remove unbalanced triangles)
  - Structural balance theorem and weak structural balance theorem

Four ways that social networks evolve

- Social networks → Triadic closure will happen
  - Person nodes with friend links → Friends of friends become friends
- Social-affiliation networks → Three kinds of closure will happen
  - Person nodes with friend links
  - Focus nodes (person's interests)
  - Triadic closure: Friends of friends become friends
  - Focal closure: Friends through common focus
  - Membership closure: Join a focus through a friend



- Friend/enemy networks
  - Strong balance: (+ + +) and (+ – –) allowed → Two enemy groups tend to form
    - (+ + –) and (– – –) will disappear
  - Weak balance: (– – –) also allowed → *n* enemy groups tend to form
    - (+ + –) will disappear

Similarity between friends in a social network

Chapter 4

Similarity between friends

- Friends in a social network tend to resemble each other
  - The principle that we resemble our friends is called homophily
    - In complex networks, this is sometimes called assortative mixing (!)
  - These similarities will influence their behavior
    - How can we take this into account?

- Links in a social network can be similar in many ways
  - Immutable characteristics: characteristics that are fixed, such as racial and ethnic groups, age, etc.
  - Mutable characteristics: characteristics that can change over time, such as occupations, affluence, beliefs, etc.

Example of homophily: racial segregation (1)

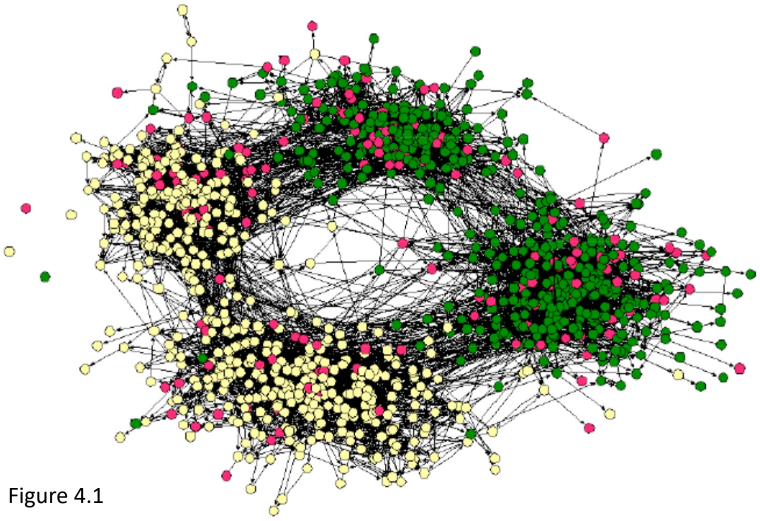


Figure 4.1

- This social network shows friendships between secondary school students in an American town
  - The network is divided into four densely connected, homogeneous parts that are weakly connected to each other
- Two divisions are apparent:
  - Race (different colored circles)
  - Age (middle and high school)

Example of homophily: racial segregation (2)

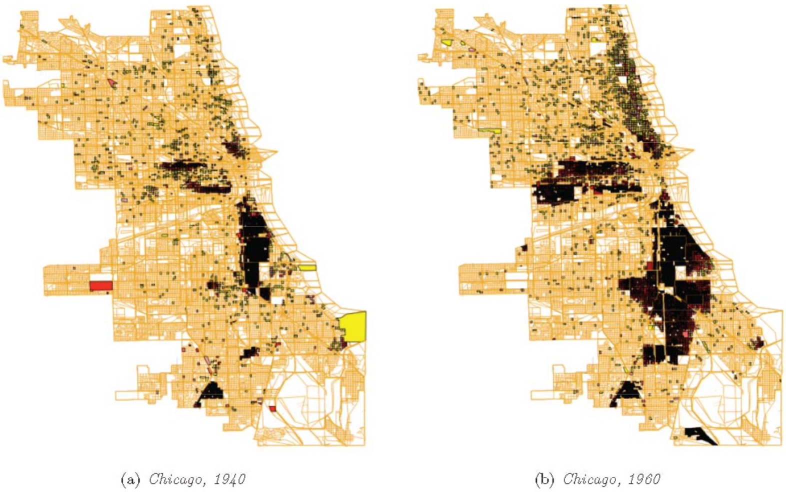

(a) *Chicago, 1940* (b) *Chicago, 1960*

Figure 4.14

- This shows maps of Chicago in 1940 and 1960
  - People choose to live in racially homogeneous neighborhoods
    - Homophily: people in the same neighborhood tend to be of the same race
  - In addition, race correlates to geography (racial segregation relates to spatial segregation)
- Yellow/orange blocks: <25% African-Americans
- Brown/black blocks: >75% African-Americans

Mechanisms underlying homophily

- There are two basic mechanisms that cause friends to resemble each other: selection and social influence (both are sociological concepts)
- Selection: people select friends with similar characteristics (an internal mechanism)
  - Individuals drive the formation of new links
- Social influence: people modify their behaviors to be closer to their friends (an external mechanism), also called “peer pressure”
  - Existing links drive the formation of new links
- If these mechanisms are outside of the graph we cannot do much
  - To study and understand them, let us bring them inside the graph!
  - We will define affiliation networks

Interplay of selection and social influence

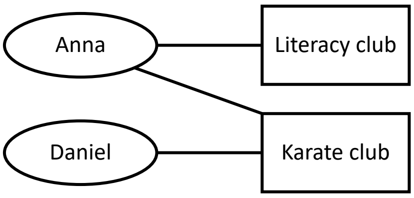
- With single snapshots of a network, it's hard to distinguish between selection and social influence
- The network must be followed over time (multiple snapshots)
  - These are called longitudinal studies
  - Track both people's social connections and people's behaviors
- Example: teenage friends tend to be similar to each other
  - Both selection (teenagers seek out people like them) and social influence (peer pressure to conform) are active
  - To influence the network, it's important to distinguish the two
    - If you make a program to reduce illicit drug use, it will only work well if the drug use is arising from social influence. Otherwise it will have little effect beyond the students it targets directly: they will change their social circles but other students are not affected!

Social-affiliation networks

Bringing homophily into the graph

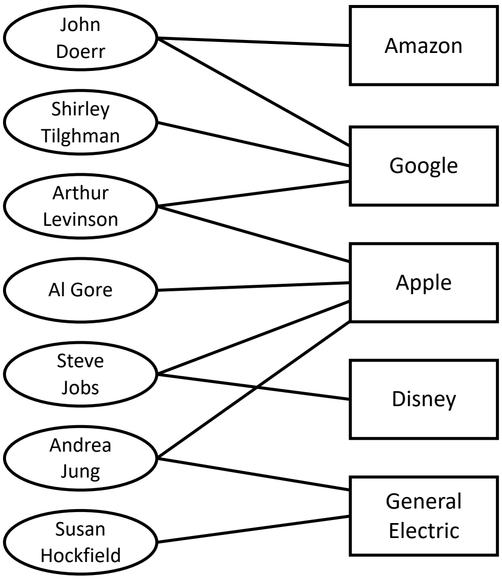
- How can we take homophily into account?
  - We would like to add similarities to the social network itself, so that we can use reasoning on graph structure!
  - We define an extended graph and use it to better understand people's behavior
  - Existence of homophily is a starting point to study the network's underlying mechanisms and the network's further evolution
    - Often, it is used to influence the people in the network!
- Rest of lecture: study different aspects of homophily
  - Mechanisms underlying homophily
  - Affiliation networks
  - Measuring homophily
  - Social-affiliation networks
  - Generalizing triadic closure
  - Real-world studies on big data sets

Affiliation networks



- An affiliation network is a bipartite graph that shows which individuals are affiliated with which activities
- Definition: a bipartite graph is a graph where the nodes can be separated into two sets such that each edge connects a node from one set to the other
  - The first set is the individuals
  - The second set is the activities, which are called foci (singular: focus)

Example of an affiliation network



John Doerr

Shirley Tilghman

Arthur Levinson

Al Gore

Steve Jobs

Andrea Jung

Susan Hockfield

Amazon

Google

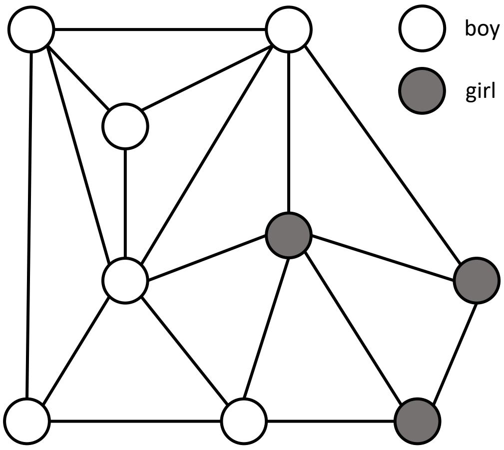
Apple

Disney

General Electric

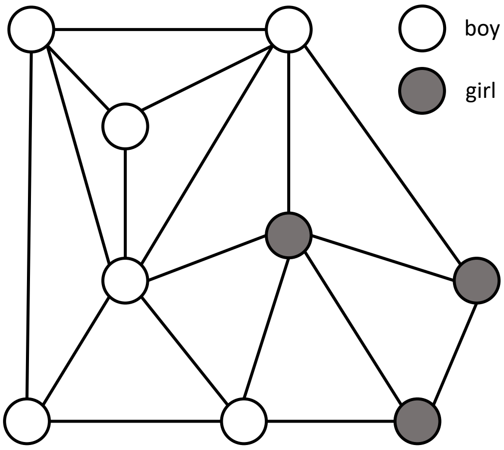
- People on corporate boards
- This is part of an affiliation network as of mid 2009
- This network can reveal important things about connections between companies!
  - For example, Steve Jobs was both in the computer and media industry: it is not surprising that Apple has moved towards media activities!

Measuring homophily (1)



- We can test whether a network exhibits homophily with respect to a given characteristic
  - Are similar nodes more likely to be connected with each other or not?
- We can do a statistical test
  - This graph shows friendships in a classroom in a primary school
  - Is there homophily?
    - Is it true that boys tend to be friends with boys, girls tend to be friends with girls?
    - We can test this

Measuring homophily (2)

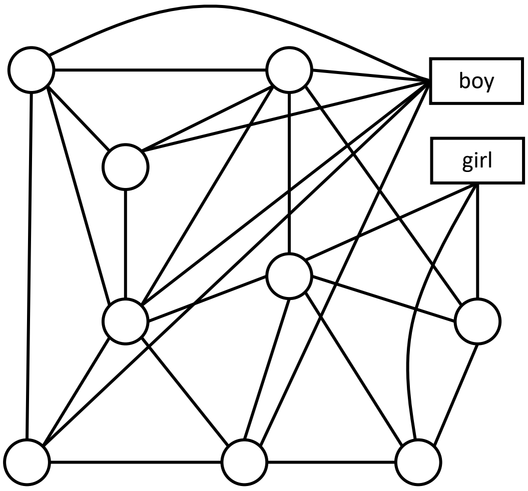


- Is there homophily in this network?
- Assume fraction p are boys and q are girls
  - If no homophily, then p^2 boy-boy edges, q^2 girl-girl edges, and 2pq boy-girl edges
  - If the fraction of boy-girl edges is significantly less than 2pq, then there is homophily
- In our example, p=2/3 and q=1/3
  - 2pq = 4/9 = 8/18
  - Actual fraction of boy-girl edges is 5/18
  - Seems to be significantly different!
- A statistical test is needed
  - Student's t-test can be used: sample mean 5/18 is within the 95% confidence interval
  - "Are the means of the populations equal?"

Social-affiliation network

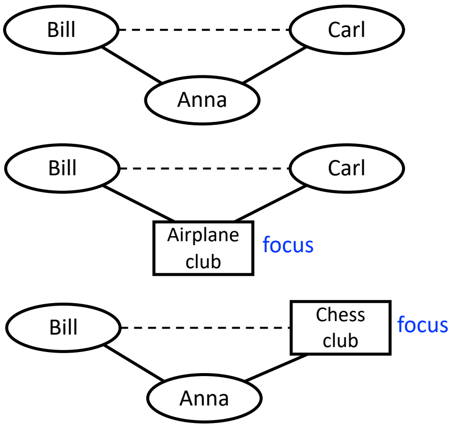
- Both social networks and affiliation networks change over time
  - New friendship links are formed and new connections with foci are formed
  - There is coevolution: two people with shared focus may become friends, two friends may influence each other's foci
- To better model this, let us combine the two networks
  - Social-affiliation network: merges social network and affiliation network
  - Two kinds of nodes: people and foci
  - Two kinds of edges: person-person edge, person-focus edge
- Let us see how social-affiliation networks evolve over time
  - Now there are three kinds of closure operations

Example of a social-affiliation network



- Our classroom example is a typical social-affiliation network
- There are two foci: boy and girl

Three forms of closure in a social-affiliation network



- Triadic closure: Bill and Carl become friends because they have common friend Anna (as we saw before)
- Focal closure: Bill and Carl become friends because they have common focus Airplane club
- Membership closure: Bill joins the Chess club because friend Anna is a member (friend and focus have common friend)

By adding foci to the network, we have brought one of the sources of social network evolution **inside the network**

Exercise on closure

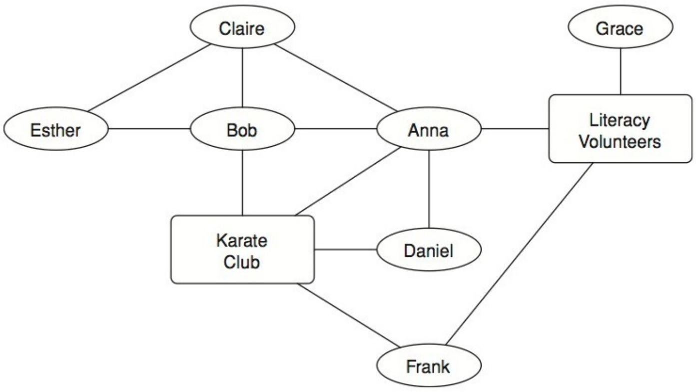


Figure 4.8. A larger network that contains the example from Figure 4.7. Pairs of people can have more than one friend (or more than one focus) in common. How does this increase the likelihood that an edge forms between them?

- Here's an example network
- With all the new information given by foci, how does this improve our knowledge of how the network evolves?
- If there are more friends or foci in common, how does this affect the likelihood that edges form?

Empirical studies of social network evolution

Empirical evidence for triadic closure

- How much more likely is an edge to form between two people if they have more friends in common?
  - Take two snapshots of a network at different times
  - For each k≥0, identify all pairs who have exactly k friends in common in the first snapshot, but are not directly connected
  - Define T(k) to be the fraction of those pairs that have connected in the second snapshot
- Kossinets and Watts have studied this question
  - Based on full history of e-mail communication among 22,000 university students at a large American university over a one-year period
  - From this, they define "friends" and "snapshots", and compute T(k): friend if communicated in the past 60 days, one snapshot per day

Kossinets and Watts study for triadic closure

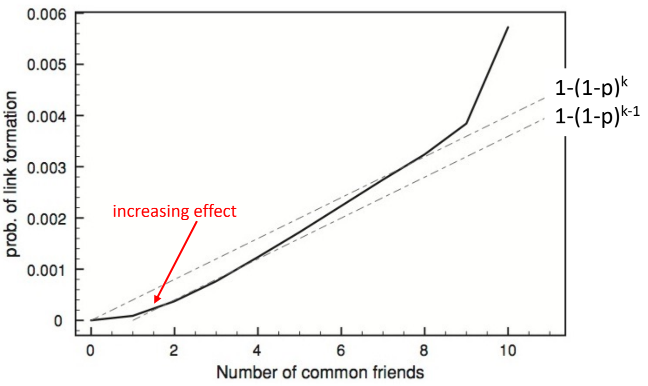


Figure 4.9. Quantifying the effects of triadic closure in an e-mail data set [259]. The curve determined from the data is shown in the solid black line; the dotted curves show a comparison to probabilities computed according to two simple baseline models in which common friends provide independent probabilities of link formation. (Image from the American Association for the Advancement of Science.)

- T(k) is average probability that two people form a link per day, as a function of number of common friends
- Evidence for triadic closure
  - Probability starts at 0 and increase in roughly linear fashion up to 8 friends, and more than linear for >8
- What about focal closure and membership closure?

Comparison to a theoretical model

- How does the linear behavior of T(k) compare to a simple model?
  - Suppose that each common friend gives an additional independent probability p of forming a link per day
- Two people with k common friends fail to link with probability $(1-p)^k$
  - Each one fails with probability (1-p) and there are k independent trials
  - Therefore the probability that at least one link forms is $1-(1-p)^k$
- This gives $T_{baseline}(k) = 1-(1-p)^k$ plotted as upper dotted line
  - $1-(1-p)^{k-1}$ is plotted as lower dotted line
- For small p, $T_{baseline}(k) \approx kp$ (linear in p)

Kossinets and Watts study for focal closure

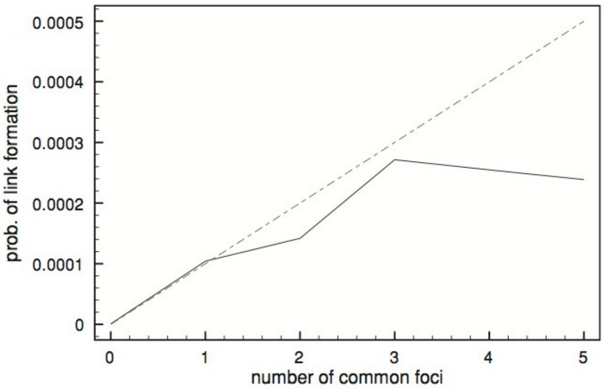
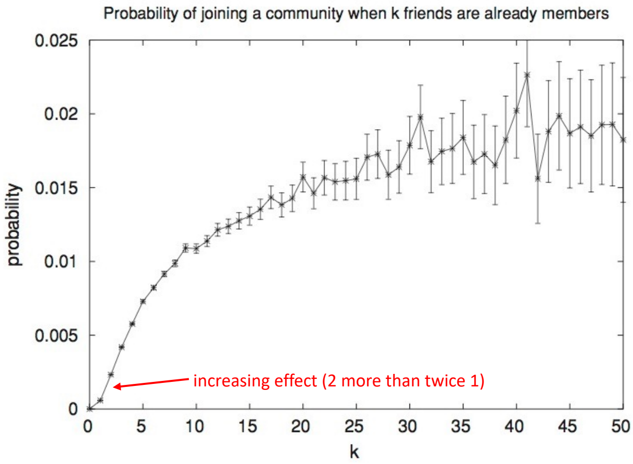


Figure 4.10. Quantifying the effects of focal closure in an e-mail data set [259]. Again, the curve determined from the data is shown as the solid black line, while the dotted curve provides a comparison to a simple baseline. (Image from the American Association for the Advancement of Science.)

- The e-mail data set was supplemented with class schedules for each student
- Each class is a focus; two students share a focus if they take a class together
- (Some) evidence for focal closure
  - Diminishing returns effect! The first class is similar to having one shared friend, but after that the curve levels off

LiveJournal study for membership closure

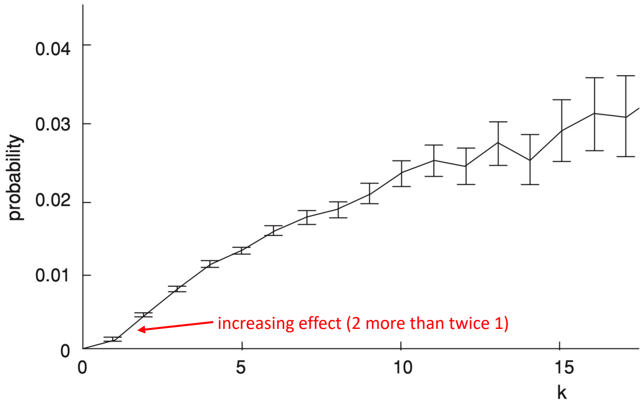


increasing effect (2 more than twice 1)

Figure 4.11. Quantifying the effects of membership closure in a large online data set: The plot shows the probability of joining a LiveJournal community as a function of the number of friends who are already members [32].

- Probability of joining a LiveJournal community as function of the number of friends who are already members
- Evidence for membership closure
  - Strong effect at first which levels off later, but the marginal effect remains strong for many friends (unlike for focal closure)
  - Increasing effect in the beginning: with 2 friends there is more than twice the likelihood (similar to triadic closure)

Wikipedia study for membership closure



increasing effect (2 more than twice 1)

Figure 4.12. Quantifying the effects of membership closure in a large online data set: The plot shows the probability of editing a Wikipedia articles as a function of the number of friends who have already done so [122].

- Probability of editing a Wikipedia article as a function of number of friends who have done so
- Evidence for membership closure
  - Flattens off less than the LiveJournal study: friend influence continues strong
  - As for LiveJournal, there is an increasing effect (2 is more than twice 1)

Comparing selection and social influence

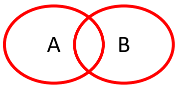
Comparing selection and social influence

- Remember the two sources of homophily
    - Selection: internal influence, choice coming from the person
    - Social influence: external influence, person adapts to their friends
- They are hard to distinguish
    - Adding focus nodes is applicable to both selection and social influence!
- Can we compare the two in a quantifiable way?
    - We need to clearly separate two parts of network evolution, one where selection predominates and another where social influence predominates
    - Seems to be very difficult!
    - Let us see how we can do it…

Wikipedia study for sources of homophily

- Wikipedia is a collaborative encyclopedia written by editors
  - We study the behavior of Wikipedia editors in terms of selection and social influence
- We define the social network
  - Two editors are linked if one has communicated on the other's talk page
  - An editor's behavior consists of the set of articles that they have edited
- We define similarity between two editors through their behaviors:

$$s(A,B) = \frac{\text{(number of articles edited by both A and B)}}{\text{(number of articles edited by at least one of A and B)}}$$



Note the resemblance to neighborhood overlap (it is in fact identical!)

Average similarity of two Wikipedia editors

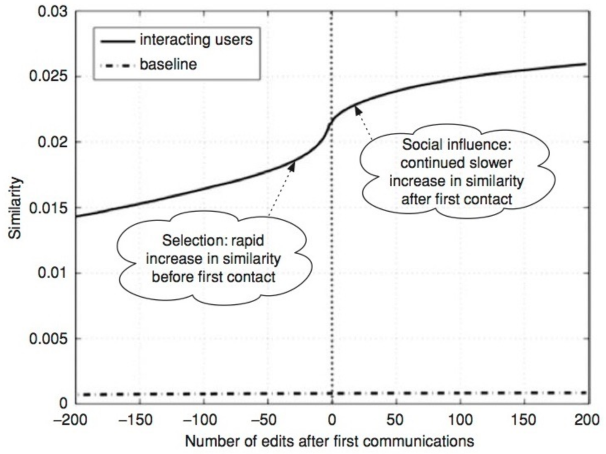


Figure 4.13. The average similarity of two editors on Wikipedia, relative to the time (0) at which they first communicated [122]. Time, on the *x*-axis, is measured in discrete units, where each unit corresponds to a single Wikipedia action taken by either of the two editors. The curve increases both before and after the first contact at time 0, indicating that both selection and social influence play a role; the increase in similarity is steepest just before time 0.

- We show average similarity of two editors relative to the instant when they first communicated
- Similarity increases faster and faster until the editors meet (they are independently becoming more similar, which shows selection)
- After they meet, similarity increase slows down as the editors adapt to each other (divide work between each other): social influence
- This curve shows averages over many editors; for individual editors it is much less smooth!

Summary of closure analysis of social networks

- These analyses show some early attempts to quantify the basic mechanisms of link formation at large scale, using real online data
- They do reveal that the closure effects are real and that prediction can give useful results
  - But there are many questions not answered
- Are the general shapes similar across different domains?
  - Including domains less linked to technology!
- Can we explain the shapes through their underlying social mechanisms?
- In conclusion, we can say that the analysis of social networks is promising but still in its early stages
  - You could extend this analysis in the course project!

Friend/enemy networks

Chapter 5

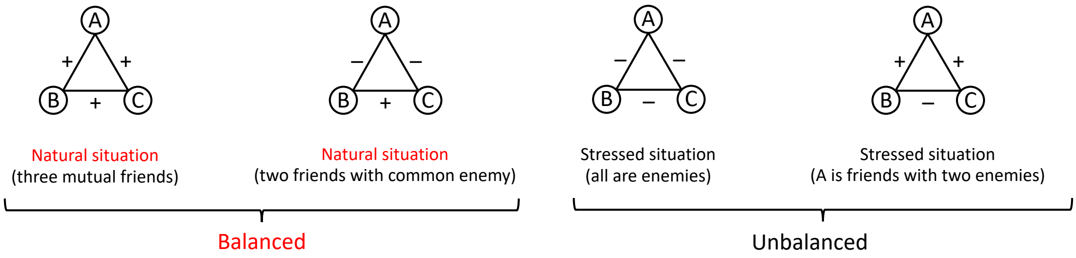
Positive and negative relationships

- Motivation
  - Up to now link strength is always positive
    - All links are positive but with different strengths (close friends or acquaintances)
  - But in many networks, there are also negative effects
    - Friend versus enemy, trust versus distrust
  - Social network becomes a friend/enemy network
- We generalize links to be positive or negative
  - We study the tension between these two forces
    - Two nodes with a common friend can become friends
    - Two nodes with a common enemy can become friends
  - We introduce the concept of structural balance
  - One consequence is that local effects can have global consequences
    - The whole network separates into friend groups that are antagonistic

Structural balance

- We consider social networks where everybody knows everybody
  - This corresponds to the definition of a clique or complete graph
    - There exists an edge joining every pair of nodes
- Each edge is labeled with + or –
  - This defines a labeled complete graph
  - No pair of people is indifferent or unaware; they are friends or enemies
  - This models a small group of people that are mutually aware
    - Classroom, small company, sports team, club, international relations
- Concept of structural balance
  - A triangle can be balanced or unbalanced
  - A graph consists of triangles, and can also be balanced or unbalanced
  - Real-world networks tend to evolve toward structural balance

Balanced and unbalanced triangles



Natural situation
(three mutual friends)

Natural situation
(two friends with common enemy)

Stressed situation
(all are enemies)

Stressed situation
(A is friends with two enemies)

Balanced

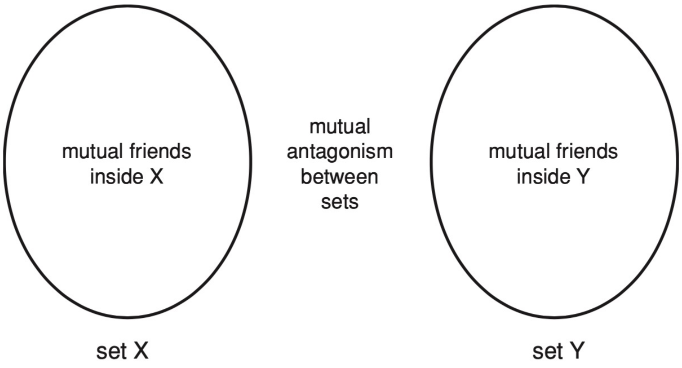
Unbalanced

- A triangle (three people) has four possible configurations
  - Two configurations are balanced (natural)
  - Two configurations are unbalanced (stressed)
- Definition: a triangle is balanced if it is annotated (+ + +) or (+ – –)
  - Otherwise it is unbalanced if it has annotations (– – –) or (+ + –)

Structural balance for networks

- Definition: a labeled complete graph is balanced if for each triangle in the graph (triple of 3 nodes), the links are (+ + +) or (+ – –)
  - Each triangle is balanced
- Example:
- This is a local definition
  - Does there exist a global definition of structural balance?

Structural balance theorem



mutual friends inside X

mutual antagonism between sets

mutual friends inside Y

set X

set Y

- Balance theorem
  [F. Harary 1953]
  If a labeled complete graph is balanced, then either all pairs of nodes are friends, or else the nodes can be divided into two groups such that nodes within a group are friends and nodes between groups are enemies
- This takes a purely local property (balanced triangle) and shows that it implies a strong global property

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Proof (1)

- Suppose we have a labeled complete graph that is balanced
  - We need to show it has the global property of the theorem
  - Be very careful that you are making the proof in this direction!
- If the graph has no negative links at all, the proof is done
  - We therefore assume there is at least one negative link
- Pick any node A of the graph
  - Let us study the graph from the viewpoint of A

Proof (2)

- We will define two sets of nodes X and Y
    - Such that each set consists of mutual friends, and the sets are enemies
    - Define set X to be A plus all of its friends
    - Define set Y to be all the enemies of A
- Do X and Y satisfy the theorem?
- We need to show the following three properties:
    i. Every two nodes in X are friends
    ii. Every two nodes in Y are friends
    iii. Every node in X is enemy of every node in Y

Proof (3)

Work out the diagram yourself!

- Let us make a diagram:
- Let us examine each of the three properties:
    i. Two nodes in X:
    ii. Two nodes in Y:
    iii. Node in X and node in Y:
- QED

Applications of structural balance

- Example 1: International relations
  - Conflict over the separation of Bangladesh and Pakistan in 1972
    - USSR-China, China-India, India-Pakistan: these pairs are enemies
    - USA wanted to improve relations with China, it supported the enemies of China's enemies, therefore it supported Pakistan
    - North Vietnam became friendly toward India
    - Pakistan severed relations with Eastern Bloc countries that recognized Bangladesh
    - China vetoed the acceptance of Bangladesh in the UN

Example 2: World War I

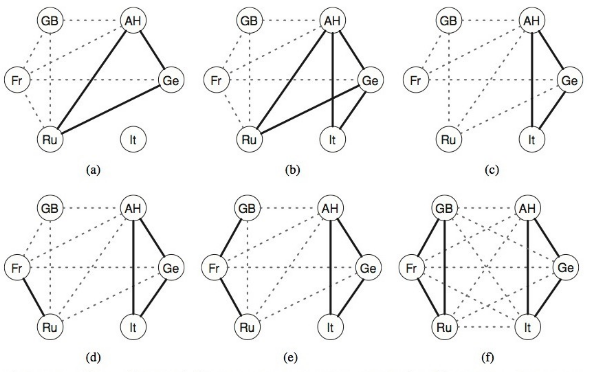


Figure 5.5. The evolution of alliances in Europe, 1872–1907 (the abbreviations GB, Fr, Ru, It, Ge, and AH stand for Great Britain, France, Russia, Italy, Germany, and Austria-Hungary, respectively): (a) Three Emperors' League, 1872–1881; (b) Triple Alliance, 1882; (c) German–Russian Lapse, 1890; (d) French–Russian Alliance, 1891–1994; (e) Entente Cordiale, 1904; (f) British Russian Alliance, 1907. Solid dark edges indicate friendship while dotted edges indicate enmity. Note how the network slides into a balanced labeling – and into World War I. (This figure and example are from Antal et al. [20] and Elsevier Science and Technology Journals.)

- Shifting alliances in Europe led toward World War I
- Structural balance is not necessarily a good thing!
  - It often leads toward two opposed alliances
  - It can sometimes be seen as the slide into a hard-to-resolve opposition

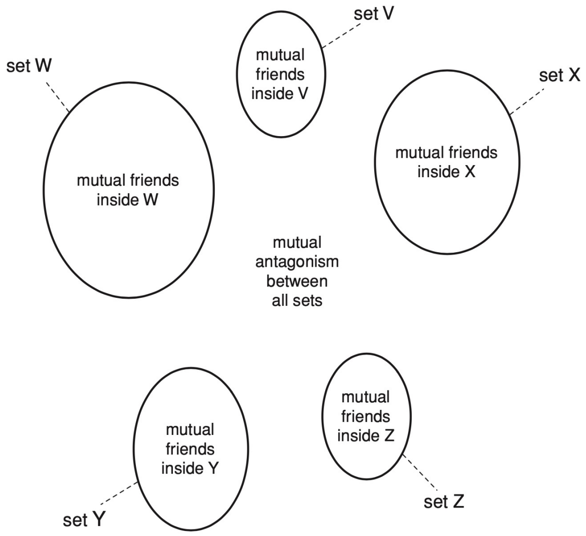
Example 3: User communities on the Web

- Assume that members can like or dislike each other
  - Trust and distrust
  - Friend and foe
- Some questions to drive the analysis (more properties of links)
  - Is the relationship commutative? Often it is, sometimes it is not.
  - Is the relationship transitive?
    - Sometimes: A trusts B, B trusts C, so A trusts C
    - Distrust is not necessarily transitive: A distrusts B, B distrusts C
      - If distrust is like being an enemy, then A trusts C is plausible
      - If distrust is like believing the other is incompetent, then A distrusts C is plausible
    - Epinions: trust-distrust models political alignment, so A trusts C is plausible
    - Website that rates products: trust-distrust models expertise, so A distrusts C is plausible
- General conclusion:
  - It is important to define clearly the basic properties of the link
  - Not all links behave like structural balance

Weak structural balance

- We can formulate a useful alternative to structural balance by modifying the original assumptions
  - We change our view of what is balance
    - (+ + –): a person with two friends who do not get along → they could get reconciled!
    - (– – –): all three are enemies, bad situation, there is much less chance of this changing
  - We can assume (– – –) triangles are mostly stable, so let’s add them
- Definition: a labeled complete graph is weakly balanced if for each triangle in the graph (triple of 3 nodes), the links are (+ + +) or (+ – –) or (– – –)
  - We allow the (– – –) triangle to exist since it is mostly stable
  - Only one disallowed case: no triangles (+ + –) because they tend to disappear

Weak structural balance theorem



- Weak balance theorem

  If a labeled complete graph is weakly balanced, then its nodes can be divided into groups such that any two nodes belonging to the same group are friends and any two nodes belonging to different groups are enemies
- This is an interesting variation of the balance theorem: *n* versus 2 groups

Proof (1)

Work out the diagram yourself!

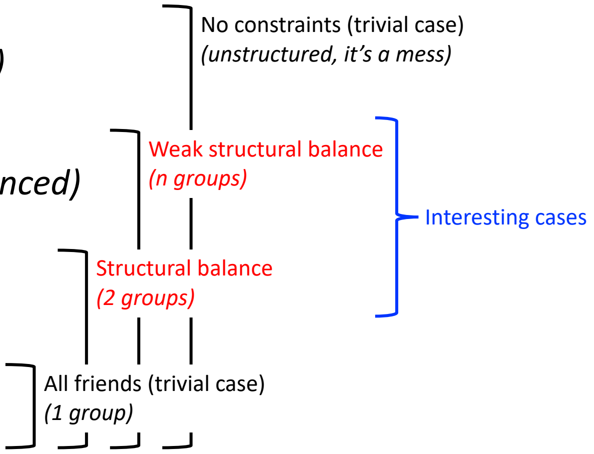
- We use the same approach as the proof of the Balance Theorem
  - How do we prove there is a division into *n* groups, where *n* is unknown?
  - This proof is very interesting because it uses recursion:
    - We construct a first group and remove its nodes, and continue until no more nodes
- Pick any node A of the graph and all its friends
  - We call this the group X
- We need to show that:
  - i. All nodes in X are mutual friends
  - ii. All nodes in X are enemies with all other nodes

Proof (2)

- Now remove X from the graph
  - Remove X’s nodes and all edges that have at least one node in X
- Look at the set of remaining nodes: it is strictly smaller than the original set
  - If this set is empty, we are done
  - Otherwise, continue the proof with this set
    - (Pick any node A of the graph and all its friends, etc.)
- Ultimately we arrive at a partition of the original graph into *n* nodes that satisfies the theorem (note that *n* is not known in advance!)
- QED

Degrees of structural balance

- (+ + –)
  (unbalanced)
- (– – –)
  (weakly balanced)
- (+ – –)
  (balanced)
- (+ + +)
  (balanced)



No constraints (trivial case)
(unstructured, it’s a mess)

Weak structural balance
(n groups)

Structural balance
(2 groups)

All friends (trivial case)
(1 group)

Interesting cases

In a real-world situation, it is important to first find the degree of balance. After that, the appropriate theorem can be applied.

Summary of structural balance

- We generalize social networks to friend/enemy networks
  - This models many real-world situations with friends and enemies
- Structural balance
  - It is a form of equilibrium: friend/enemy networks tend to evolve toward balance
  - Balance theorem shows that this implies two enemy groups
  - This has some interesting applications (World War I)
  - This can be generalized by adding more properties to the edges
    - Commutativity or lack of it, transitivity or anti-transitivity or lack of it
- Weak structural balance
  - It is another form of equilibrium where more antagonism is allowed
  - Weak balance theorem shows that this implies *n* enemy groups
  - This also has interesting applications (Tour de France)

LINFO1115
Reasoning about a highly connected world

Lectures 3 and 4
Introduction to game theory

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The two foundations of the course

Last week

- Graph theory
  - Social networks as graphs: combining graph theory (mathematics) + sociology (humanities)
  - Social-affiliation networks: formalizing reasons for homophily (focus nodes)
  - Strong and weak links, graph evolution toward closure (three kinds)
  - Positive and negative links, graph evolution toward balance (weak balance and strong balance)

This week
and next week

- Game theory
  - Social networks with interactions: combining game theory + sociology
  - Games: formalizing interactions between participants
  - First part: simple games with two rational players
    - Rational behavior: strictly dominant strategies
    - Nash equilibrium: what to do when there is no dominant strategy
    - Kinds of games: coordination and anticoordination games
    - Mixed strategies: when there is no Nash equilibrium for pure strategies
  - Second part: extensions
    - Pareto optimality and social optimality: maximizing for society, not individual players
    - Dynamic games: sequence of moves, not just one simultaneous move

Introduction to game theory

Chapter 6

Games: social networks with interactions

- Previously, each person in a social network acted independently
  - Using what they know, who are their friends and their other knowledge (foci)
  - We have introduced link strength and affiliation to model this knowledge
- But in the real world, people interact!
  - Their decisions influence each other
  - They will decide depending on how they perceive the other will decide
  - They will reason about what is their best decision w.r.t. the other's decisions
- Game theory studies players' interactions
  - The participants are part of a "game", an interaction defined by a set of rules
  - Each participant has a goal and uses some notion of rational behavior

We will see different kinds of rationality!

Basic game theory and extensions

- Part 1: Games with two rational players
  - Two opposing players make simultaneous decisions and optimize their gains
  - Also called “rational games” since each player makes “rational decisions”
  - Each player knows the payoff table and can compute their best strategy
- Part 2: Extensions
  - Games with global optimality: achieving results good for society
    - Pareto optimality or social optimality
  - Games played over time (dynamic games)
    - Sequence of moves observed by the players
- Rest of course: Games on networks, with more players
  - Traffic, auctions, markets, intermediaries, negotiation, cascades

Part 1:
Games with two rational players

Overview

- Games with two players and simple rules
  - We present some simple games to introduce the key ideas
  - We show some strategies: a strategy is a set of options for playing a game
- Best responses and dominant strategies
  - We study a player's best response, given a belief about the other player
  - A dominant strategy is a best response to every strategy of the other player
- Nash equilibrium: what to do when you have no dominant strategy
  - Find a Nash equilibrium: each has a strategy that is best response to the other
- Multiple equilibria
  - Coordination games (real goal is to work together), anticoordination games
- No equilibria
  - Mixed strategies where the players randomize their behavior

A simple game redux

		Your partner	
		Presentation	*Study exam*
You	*Presentation*	90, 90	86, 92
	Study exam	92, 86	88, 88

I prepare the presentation.
Partner studies for exam.
I get (80 + 92)/2 = 86%
Partner gets (92 + 92)/2 = 92%

- Remember the exam-or-presentation game we introduced during the first lecture
- You and your partner are students
  - Tomorrow each has a joint presentation and an individual exam: study exam or prepare presentation?
  - Exam: if you study, you get 92%, if not you get 80%
  - Presentation: if both prepare ⇒ both 100%; one prepares ⇒ both 92%; none prepares ⇒ both 84%
- Total grade is average of exam and presentation
- Simple reasoning shows that each partner's strategy should be to study for the exam!
  - But if they both prepare for the presentation, they will both get more points!
  - What is going on here? Let us study this closer!

Definition of a game

- To study this better, we first define the concept of a game
- A game is any situation satisfying the following three conditions:
  - There is a set of participants, called the players. In this example, there are two players.
  - Each player has a set of options for how to behave: we will refer to these options as the player's strategies. In this example, you and your partner each have two possible strategies: prepare for presentation or study for exam.
  - For each choice of strategies (one by each player), each player receives a payoff. The payoff depends on the strategies selected by all players. Payoffs are generally numbers, and each player prefers larger payoffs to smaller ones. In this example, the payoff is the average grade for exam and presentation.
- We reason about how the players will behave in a given game
  - We start with games with two players that play only once and where the players simultaneously and independently choose their actions (one-shot games)
  - We later extend this for games that are played sequentially over time (dynamic games)

Underlying assumptions of a game

- The underlying assumptions must be chosen very carefully
  - What is a best strategy depends critically on these assumptions

- Assumption 1: all that a player cares about is summarized in the payoff
  - If the player is "altruistic" and cares about the other, it must be in the payoff!
- Assumption 2: each player knows everything about the game
  - This is called common knowledge: they know the game, they know that each of them knows the game, they know that each of them knows that each of them knows, and so on (to infinity)
  - If not, it leads to the theory of games with incomplete information (more complicated!)
- Assumption 3: each player chooses a strategy to maximize their own payoff, given a belief about the strategy used by the other player
  - This is called rationality. It actually combines two ideas: (1) each player wants to maximize payoff, and (2) each player actually succeeds in selecting the optimal strategy. In complex games this is not always possible and players sometimes make mistakes.

Reasoning in the exam-or-presentation game

- Let us reason about our example game by following the assumptions
- Focus on your own choices first: what is your best strategy?
    - If you knew the partner would study, you would get payoff of 88 by studying and 86 by preparing the presentation. In this case, you should study.
    - If you knew the partner would prepare the presentation, you would get a payoff of 92 by studying and 90 by preparing the presentation. In this case, you should study.
- When a player has a strategy strictly better than all others, regardless of what the other player does, it is called a strictly dominant strategy
    - Here, the strictly dominant strategy is to study in all cases, with grade 88 for each
    - So the analysis is clean and the outcome is simple!
    - But there is something strange: if both would prepare, they would each get 90
    - What is wrong?

Rationality depends on the assumptions

- What is going on here?
    - The payoff of 90 cannot be reached by rational play, in the way that rationality is defined here
    - It is because even if you were to commit to preparing the presentation, hoping to get 90, and even if your partner knew this, your partner would still have an incentive to study to get a higher payoff of 92
- The result depends on the assumption that payoffs truly reflect everything each player values in the outcome
    - If you would care about your partner's grade as well as yours, it would be in the payoff!
    - This shows the importance of choosing the correct assumptions. Once the assumptions are fixed, rationality takes its course.
    - Changing how the payoff is calculated, for example taking into account that your partner would be angry if you did not prepare the presentation, would affect the results

Strictly dominant strategies

Prisoner’s Dilemma

		Suspect 2	
		Not-Confess	*Confess*
Suspect 1	*Not-Confess*	-1, -1	-10, 0
	Confess	0, -10	-4, -4

- The exam-or-presentation game is closely related to a famous example, the Prisoner’s Dilemma
  - Two suspects are interrogated in independent rooms
  - Both could be charged with resisting arrest and get a 1-year sentence
  - If one suspect confesses and the other does not, the confessor will be released and the other will get a 10-year sentence
  - If both confess, both will get a 4-year sentence
  - If neither confesses, both will get a mild 1-year sentence
- If you are Suspect 1, do you confess or not?

Reasoning for the Prisoner’s Dilemma

- We start by reasoning about Suspect 1’s options:
  - If Suspect 2 would confess, then Suspect 1 would get 4 years by confessing and 10 years by not confessing. So Suspect 1 should confess.
  - If Suspect 2 would not confess, then Suspect 1 would be released by confessing and get 1 year by not confessing. So Suspect 1 should confess.
- So confessing is a strictly dominant strategy: it is the best choice regardless of what the other player chooses
  - Rationality, i.e., strictly following our assumptions, therefore leads to both suspects confessing
  - They both get 4-year sentences, but if they would both stay silent they would only get 1 year!
  - Again, rationality leads to an outcome that is worse for both

Interpreting the Prisoner’s Dilemma

- The Prisoner’s Dilemma is a streamlined depiction of the difficulty to establish collaboration in face of individual self-interest
  - It has been hugely studied since its introduction in the 1950s!
  - It has been used as a framework for many real-world situations
  - These situations are called arms races, where two competitors increasingly use dangerous options simply to remain evenly matched
- Another example: use of performance-enhancing drugs
  - We assume that drugs are difficult to detect
  - In this case, both athletes will choose to use drugs, even though not using is better (no long-term harm, cannot get caught)

		Athlete 2	
		Don’t Use Drugs	*Use Drugs*
Athlete 1	*Don’t Use Drugs*	3, 3	1, 4
	Use Drugs	4, 1	2, 2

Avoiding the Prisoner’s Dilemma

- There are several ways to avoid the Prisoner’s Dilemma
  - One way is to calculate the payoff to take collaboration into account (if the players can accept that this may reduce their personal gains!)
  - Another way is to change the payoff values, since the Dilemma only arises if the payoffs are aligned in a certain way

Your partner

		Presentation	*Exam*
You	*Presentation*	98, 98	94, 96
	Exam	96, 94	92, 92

- The exam-or-presentation example now has presentation as the strictly dominant strategy
  - We did this by changing the payoff to make the exam easier: 100 if you study and 96 if you don’t

Best response

- Let’s formalize what we saw so far
  - Assume S is the strategy chosen by Player 1 and T is the strategy chosen by Player 2
  - The pair of strategies (S,T) corresponds to an entry in the payoff matrix
  - $P_1(S,T)$ is payoff to Player 1 as a result of (S,T)
  - $P_2(S,T)$ is payoff to Player 2 as a result of (S,T)

Player 2

			T
Player 1			
	S		(P_1 , P_2)

- Definitions:
  - S for Player 1 is best response to T for Player 2 if:
    - For all other strategies S’ of Player 1: $P_1(S,T) \geq P_1(S',T)$
  - S for Player 1 is strict best response to T for Player 2 if:
    - For all other strategies S’ of Player 1: $P_1(S,T) > P_1(S',T)$

		T	
S’		$(P_1' , \_)$	$P_1 > P_1'$
S		$(P_1 , \_)$	

Strictly dominant strategy

- Does any player have one strategy that is always strictly better?
  - One strategy that is a strict best response to every strategy of the other player
  - Such a strategy, if it exists, is called a strictly dominant strategy
- In the Prisoner's Dilemma, both players have a strictly dominant strategy
  - This makes the analysis simple: both players will use their strictly dominant strategy
  - What about the exam-or-presentation game?
- Most games are not so simple
  - Maybe a player has a dominant strategy that is not strict (it is tied with others)
  - Maybe only one player has a strictly dominant strategy
  - Maybe no players have dominant strategies

One strictly dominant strategy

- Here's a game where only one player has a strictly dominant strategy
- Firm 1 and Firm 2 each plan to introduce a product
  - Each firm has two strategies: a low-price product or an upscale product
  - Two kinds of consumers: 60% only buy low-price and 40% only buy upscale
  - When Firm 1 and Firm 2 compete directly, Firm 1 gets 80% of sales
  - Profit is the same for selling one low-price product and one upscale product

		Firm 2	
		Low-Priced	*Upscale*
Firm 1	*Low-Priced*	0.48, 0.12	0.60, 0.40
	Upscale	0.40, 0.60	0.32, 0.08

- Firm 1 has a strictly dominant strategy: make a low-priced product
- Firm 2 does not have such a strategy. What should they do? They should react to Firm 1's strategy.
- Prediction: Firm 1 makes 0.60, Firm 2 makes 0.40
- Intuition: Firm 1 is so strong it can ignore Firm 2, whereas Firm 2 must stay safely away from Firm 1.

Nash equilibrium

Nash equilibrium

- What happens when neither player has a strictly dominant strategy?
    - We need some other way to predict what the players will do!
- The Nash equilibrium is one way out
    - Nash equilibrium: each player’s strategy is a best response to the other
    - This is an equilibrium since neither has an incentive to deviate
    - This solves the problem when a game has exactly one Nash equilibrium
- Multiple Nash equilibria
    - Sometimes there are multiple Nash equilibria
    - How does a player choose between the two?
    - We look at different ways to distinguish between the Nash equilibria

Two firms with three clients (1)

			Firm 2	
		A	B	C
	A	4, 4	0, 2	0, 2
Firm 1	B	0, 0	1, 1	0, 2
	C	0, 0	0, 2	1, 1

- Let’s study a game that has no strictly dominant strategies
- It is a marketing game between two firms (like the previous example)
  - But in this game, there are three large clients A, B, C
  - Each firm has three possible strategies: to approach client A, B, or C
- Rules:
  - A is big and worth 8, whereas B and C are smaller and worth 2 each
  - If the two firms approach the same client, they each get half the business
  - Firm 1 is too small to attract business on its own: without Firm 2, it gets zero
  - If Firm 2 approaches B or C, it gets full business. But A requires both firms together.

Two firms with three clients (2)

			Firm 2	
		A	B	C
	A	4, 4	0, 2	0, 2
Firm 1	B	0, 0	1, 1	0, 2
	C	0, 0	0, 2	1, 1

- Neither firm has a dominant strategy
  - How do we work this out?
- Each strategy by each firm is a strict best response to some strategy by the other firm
  - For Firm 1: A is strict best response to A, B to B, and C to C
  - For Firm 2: A is strict best response to A, B to C, and C to B
- How do we reason about this game?

Defining the Nash equilibrium

- Even when there are no dominant strategies, we can expect players to use strategies that are best responses to each other
  - Principle proposed by John Nash in 1950; he received the Nobel Prize in 1994
- Suppose that Player 1 chooses S and Player 2 chooses T
  - Definition: We say that (S,T) is a Nash equilibrium if S is best response to T and T is best response to S
- This concept is not deducible from rational play of the players; rather it is an equilibrium concept: no player has an incentive to deviate
  - Why not? Assume that the strategies are not best responses. Then at least one player would have an incentive to change strategy.
  - If each player believes that the other will play one part of a Nash equilibrium, then they have an incentive to play the other part of the Nash equilibrium

Two firms with three clients (3)

		Firm 2		
		A	B	C
Firm 1	A	4, 4	0, 2	0, 2
	B	0, 0	1, 1	0, 2
	C	0, 0	0, 2	1, 1

- Let us consider this game from the viewpoint of Nash equilibrium
- If Firm 1 chooses A and Firm 2 chooses A, then we can check that Firm 1 is playing a best response to Firm 2, and also that Firm 2 is playing a best response to Firm 1
  - The pair (A,A) forms a Nash equilibrium
  - This is the only Nash equilibrium: no other pair are best responses to each other (note: later we will find other equilibria when we look at mixed strategies)

Multiple Nash equilibria: coordination games

Multiple Nash equilibria

- Some natural games can have more than one Nash equilibrium
  - How do we predict what rational players will do in such games?
  - The problem is how to choose between the multiple Nash equilibria
- We will look at two kinds of games with multiple equilibria
  - Coordination games where the goal is to work together
  - "Anticoordination" games where the goal is to compete
- Coordination games
  - Nash equilibria correspond to collaboration where both players gain
- "Anticoordination" games
  - Nash equilibria correspond to one player gaining and the other losing

A coordination game (1)

Your partner

		PowerPoint	*Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	1, 1

- Suppose you and your partner are working on a joint presentation
  - You can’t reach your partner and you need to start working now
  - You have to decide whether to use PowerPoint or Keynote
- This is a coordination game: the real goal is to coordinate on the same strategy
  - Coordination games are very common: two manufacturers must decide whether to use metric or English units, two platoons in the army must decide whether to attack on the left or on the right, two people trying to find each other in a crowded store must decide whether to go north or south
  - Either choice is fine, but both choices must be the same!

A coordination game (2)

Your partner

		PowerPoint	*Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	1, 1

- The problem is that this game has two Nash equilibria
  - (PowerPoint,PowerPoint) and (Keynote,Keynote)
  - What to do? We need a new concept.
- Focal point: natural reasons to focus on one of the Nash equilibria
  - Two drivers approaching at night on an undivided country road. Should each driver move to the right or the left? Social convention on continental Europe says they should move to the right (in the UK it would be to the left).

Unbalanced coordination game

		Your partner	
		PowerPoint	*Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	2, 2

- Suppose that each player prefers Keynote to PowerPoint
  - You still want to coordinate, but now the two alternatives are unequal
- The game has the same two Nash equilibria
  - Despite that one of them gives higher payoffs to both players
  - The theory of focal points suggests to use a feature intrinsic to the game (rather than an arbitrary social convention) to predict the chosen equilibrium
  - We can predict that the players will select strategies that give higher payoffs

“Battle of the sexes”

		Your partner	
		PowerPoint	*Keynote*
You	*PowerPoint*	1, 2	0, 0
	Keynote	0, 0	2, 1

- Assume the players don’t agree on which software is preferred
  - This is a third variation on the coordination game
  - There are still two Nash equilibria, but each one favors another player
- A husband and wife want to see a movie together
  - They can choose a romantic comedy or an action movie
  - (Romance,Romance) gives higher payoff to one,
    whereas (Action,Action) gives higher payoff to the other

Stag Hunt game (1)

Hunter 2

Hunter 1		*Hunt stag*	*Hunt hare*
	Hunt stag	4, 4	0, 3
	Hunt hare	3, 0	3, 3

- This is a fourth variation on the coordination game
  - The story comes from the writings of Jean-Jacques Rousseau
- Two hunters are in a forest
  - If they work together, they can catch a stag
  - On their own, they can each catch a hare
  - If one hunter tries to catch a stag alone, the result is nothing!

Stag Hunt game (2)

Hunter 2

Hunter 1		*Hunt stag*	*Hunt hare*
	Hunt stag	4, 4	0, 3
	Hunt hare	3, 0	3, 3

- Stag Hunt is also a coordination game with two Nash equilibria (Stag,Stag) and (Hare,Hare)
  - But if the players miscoordinate, the one aiming high gets penalized more
- Stag Hunt is similar in some ways to Prisoner’s Dilemma
  - Both share the property that players benefit if they cooperate but risk suffering if they try cooperating while the partner does not
  - Original exam-or-presentation game was similar to Prisoner’s Dilemma; with small change it becomes Stag Hunt
    - If both prepare, both get 100 on the presentation, if at most one prepares, both get the base grade of 84

Exam-or-presentation
(Stag Hunt version)

	Presentation	*Exam*
Presentation	90, 90	82, 88
Exam	88, 82	88, 88

Multiple Nash equilibria: “anticoordination” games

Hawk-Dove game (1)

		Animal 2	
		Dove (passive)	*Hawk (aggressive)*
Animal 1	*Dove*	3, 3	1, 5
	Hawk	5, 1	0, 0

- Two animals have a contest how to divide a piece of food
  - Each animal can choose to behave aggressively (Hawk) or passively (Dove)
  - If both are passive, they divide the food evenly
  - If only one is aggressive, it gets most of the food
  - If both are aggressive, nobody gets the food (it is destroyed or stolen)

Hawk-Dove game (2)

Animal 2

		Dove (passive)	*Hawk (aggressive)*
Animal 1	*Dove*	3, 3	1, 5
	Hawk	5, 1	0, 0

- The Hawk-Dove game has two Nash equilibria: (Dove,Hawk) and (Hawk,Dove)
  - Without knowing more, we cannot predict which of these equilibria will be played
  - Using Nash equilibrium narrows down the predictions, but does not provide a unique solution
- This game has many applications
  - Two countries choosing whether to be aggressive or passive in their foreign policy
    - If they are both aggressive, they risk going to war, which would be disastrous for both!
  - In equilibrium we can predict one will be aggressive and one will be passive, but not which one

Exam-or-presentation (Hawk-Dove version)

- The original exam-or-presentation game was like Prisoner's Dilemma
  - If both players follow their strictly dominant strategy, they get less!
- With a small change it became similar to Stag Hunt
  - There is no strictly dominant strategy, coordination is rewarded and miscoordination is punished (the one aiming higher is punished more)
- With another small change it becomes similar to Hawk-Dove
  - Now, passive means preparing for the presentation and aggressive means preparing for the exam
  - If neither prepares, there is a very low joint grade of 60
  - If both try to avoid the passive role, payoff is very low

Exam-or-presentation
(Hawk-Dove version)

	Presentation	*Exam*
Presentation	90, 90	86, 92
Exam	92, 86	76, 76

Mixed strategies

Generalizing what is a strategy

- In the previous sections, we studied games with multiple equilibria
  - The complexity comes from the existence of more than one equilibrium
- There also exist games with no Nash equilibria
  - For these games we enlarge possible strategies: we add the possibility of randomization
- If players are allowed to behave randomly, then Nash equilibria always exist
  - A random choice between two pure strategies is called a mixed strategy

- “Attack-defense” games benefit from mixed strategies
  - One player is the attacker, the other is the defender
  - Attacker uses strategies A and B, defender uses “defend against A” or “defend against B”
  - If the defender defends against the same attack, then the defender gets higher payoff
  - If the defender defends against the wrong attack, then the attacker gets higher payoff

Matching Pennies game (1)

Player 2

Player 1

	Heads	Tails
Heads	−1, +1	+1, −1
Tails	+1, −1	−1, +1

- Matching Pennies is a simple attack-defense game
  - Two people each hold a penny and simultaneously show Heads or Tails
  - Player 1 loses the penny if they match and wins the penny if they don't match
- This is a zero-sum game: the payoffs sum to zero in every outcome
  - Many attack-defense games are zero-sum games
- A similar example is the Allied landing in Europe on June 6, 1944
  - Allies: land at Normandy or Calais, German army: defend Normandy or Calais

Matching Pennies game (2)

Player 2

Player 1

	Heads	Tails
Heads	−1, +1	+1, −1
Tails	+1, −1	−1, +1

- There is no pair of strategies that are best responses to each other
  - For any such pair, one player would get –1 and would do better by switching!
- There is no Nash equilibrium for this game
  - What should the players do?
  - In real life, players try to make it difficult for opponents to predict their behavior
  - So we should try randomizing behavior between Heads and Tails!

Introducing randomized behavior

- For Matched Pennies, we can introduce randomized behavior
- The simplest way is to say that the player will not choose Heads or Tails directly, but chooses the **probability** for choosing Heads or Tails
- Strategies for Player 1 are real numbers p between 0 and 1
    - Number p means choose Heads with probability p, Tails with probability 1-p
- Strategies for Player 2 are real numbers q between 0 and 1
- By allowing randomization, we have changed the game!
    - It no longer consists of two strategies by each player, but has a set of strategies
    - We say "pure strategy" for Player 1 when the value of p is 0 or 1 (and also for Player 2 when q is 0 or 1)

Payoffs with mixed strategies (1)

- Payoffs are now random quantities (+1 / –1 with some probabilities)!
- We need a principled way to compute payoffs
- Consider Player 1
    - If Player 2 chooses strategy q it means H with prob. q and T with prob. 1-q
    - If Player 1 chooses strategy H it means –1 with prob. q and +1 with prob. 1–q
    - Player 1's expected payoff is –q + (1–q): it is already a random quantity!
- What is more appealing to Player 1: H or T?
    - Expected payoff if Player 1 plays H: (–1)(q) + (+1)(1–q) = 1–2q
    - Expected payoff if Player 1 plays T: (+1)(q) + (–1)(1-q) = 2q–1

Payoffs with mixed strategies (2)

- To reason about how each player will maximize payoff, we need an additional assumption
- Assumption M: players rank distributions over payoffs according to their expected values
- Mixed-strategy version of Matching Pennies game:
  - Strategies are probabilities of playing H
  - Payoffs are the expectations of the payoffs from the four pure outcomes (H,H), (H,T), (T,H), (T,T)
- Is there a Nash equilibrium for this richer version of the game?

Equilibrium with mixed strategies

- We observe that no pure strategy can be part of a Nash equilibrium
  - Suppose that pure strategy H by Player 1 would be part of a Nash equilibrium
  - Then Player 2's unique best response would be H as well (with +1 payoff)
  - But H by Player 1 is not a best response to H by player 2!
- What is Player 1's best response to strategy q used by Player 2?
  - Pure strategy H would give expected payoff 1–2q
  - Pure strategy T would give expected payoff 2q–1
- Key idea
  - If 1–2q ≠ 2q–1 then one of the pure strategies H or T is the unique best response by Player 1 to strategy q used by Player 2
  - Simply because one of the two is larger! But this cannot be a Nash equilibrium.

Why not?

Nash equilibrium for Matching Pennies

- We have shown that in any Nash equilibrium for mixed-strategy Matching Pennies, we must have:
  - 1–2q = 2q–1
- Solving this gives:
  - q = 1/2
  - p = 1/2 (by symmetry)
- Conclusion:
  - The only possible Nash equilibrium is (p=1/2, q=1/2)
  - We can verify that these strategies are indeed best responses to each other, so this is indeed a Nash equilibrium (exercise: verify it!)

The indifference principle

The concept of indifference

- If Player 1 believes that Player 2 will play H more than 1/2 the time
  - Then Player 1 should play T!
  - And conversely!
  - Bias towards H or T from Player 2 leads to bias towards T or H from Player 1
- Choice of q=1/2 by Player 2 makes Player 1 indifferent to H or T
  - Strategy q=1/2 is “nonexploitable” by Player 1
  - It’s because Player 2 is unpredictable and Player 1 can’t take advantage
- The concept of indifference is a general principle in similar games
  - Each player should randomize to make the other player indifferent
  - It can be generalized to games with any finite number of players and any finite number of strategies: every such game has at least one mixed-strategy equilibrium

Some real-world situations

- In some sports the players actively randomize their actions!
  - In tennis: serve up the center or to a corner
  - In soccer: penalty kick to left or right
  - Card games: to bluff or not to bluff
  - Rock-paper-scissors: pick one randomly
- A mixed strategy can be a proportion in a population
  - Two animal species where one always attacks and the other defends
  - Each individual animal is hardwired to always be aggressive or passive
  - The whole population is in a kind of mixed equilibrium, even though each individual is playing a pure strategy
  - (For further study: there is a link with evolutionary biology, see Chapter 7)

Run-Pass game (1)

Defense

		Defend Pass	*Defend Run*
Offense	*Pass*	0, 0	10, –10
	Run	5, –5	0, 0

- The Run-Pass game is a situation in American football
    - Two teams are on the field in a particular position in between the two goals, the offensive team has the ball and the other is defending
    - They prepare a play: the offensive team wants to advance toward their goal, the defensive team wants to prevent them from advancing
    - If the defense is correct, the advance is 0 yards
    - If the defense is wrong, the offense advances (either 10 yards or 5 yards)

Run-Pass game (2)

Defense

		Defend Pass	*Defend Run*
Offense	*Pass*	0, 0	10, –10
	Run	5, –5	0, 0

- There is no Nash equilibrium with a pure strategy (check it!)
    - So both teams need to use mixed strategies
- Let’s work out the Nash equilibrium
    - p is the probability that the offense passes
    - q is the probability that the defense defends against the pass
- We use the principle of indifference

Run-Pass game (3)

		Defense	
		Defend Pass	*Defend Run*
Offense	*Pass*	0, 0	10, –10
	Run	5, –5	0, 0

- Assume defense chooses probability q for defending against a pass
  - Offense payoff for passing = (0)(q) + (10)(1–q) = 10–10q
  - Offense payoff for running = (5)(q) + (0)(1–q) = 5q
  - Indifference principle says 10–10q=5q which means q=2/3
- Assume offense chooses probability p for passing
  - Defense payoff for defending against pass = (0)(p) + (–5)(1–p) = 5p–5
  - Defense payoff for defending against run = (–10)(p) + (0)(1–p) = –10p
  - Indifference principle says 5p–5=–10p which means p=1/3

What is expected payoff for offense and defense?

Strategic interpretation

- Passing, the offense’s most powerful weapon, is used only 1/3 time
  - This seems counterintuitive: why not use more your more powerful option?
- The calculation gives the answer
  - If the offense would pass more often, the defense’s best option would be to always defend against the pass, and offense would actually do worse!
- Try p=1/2 for the offense (pass more often)
  - The defense will always defend against the pass, so the offense’s expected payoff will be (1/2)(0) + (1/2)(5) = 5/2 = 2.5
  - The offense’s expected payoff for the mixed strategy is 10/3 = 3.333 (better!)
- Another way of thinking: the defense defends against the pass 2/3 of the time, even though the offense only uses it 1/3 of the time

Why *always*?

Penalty-Kick game (1)

Keeper

		Dive left	*Dive right*
Kicker	*Kick left*	0.58, –0.58	0.95, –0.95
	Kick right	0.93, –0.93	0.70, –0.70

- We model penalty kicks in soccer as a two-player game
  - Results from analysis of Ignacio Palacios-Huerta (2002) of 1400 penalty kicks
- Kicker can aim left or right, keeper can dive left or right
  - The center is very rarely used, we will ignore center strategies!
  - We factor out the kicker's left or right footedness (right = "natural side")

Penalty-Kick game (2)

Keeper

		Dive left	*Dive right*
Kicker	*Kick left*	0.58, –0.58	0.95, –0.95
	Kick right	0.93, –0.93	0.70, –0.70

- We find the Nash equilibrium using the principle of indifference
  - q is the probability that the keeper dives left
  - Kicker must be indifferent: (0.58)(q)+(0.95)(1–q) = (0.93)(q)+(0.70)(1–q)
  - This gives q=0.42 (analogous calculation for p gives p=0.39)
- This is almost exactly the result of real penalty kicks
  - Real kicks give q=0.42 and p=0.40
  - Nice to see validation of theory in a real game played by experts!

Finding all Nash equilibria

Finding all Nash equilibria

- Let’s see how to find all equilibria in a two-player two-strategy game
  - A game can have both pure and mixed-strategy equilibria
- First check first all four pure outcomes (pairs of pure strategies)
- Then check for p, q that are best responses to each other
  - We find Player 2’s strategy (q) from the requirement that Player 1 randomizes
  - Player 1 will only randomize if the pure strategies have equal payoffs
    - This gives an equation for determining q
  - Conversely, we can find p
  - If both p and q are strictly between 0 and 1, then we have a mixed-strategy equilibrium

Equilibria in unbalanced coordination game

Your partner (Player 2)

		PowerPoint	*Keynote*
You (Player 1)	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	2, 2

- Let’s consider the unbalanced coordination game
- Assume Player 2 chooses q for PowerPoint strictly between 0 and 1
  - Player 1 will be indifferent if $(1)(q)+(0)(1–q) = (0)(q)+(2)(1–q)$ giving $q=2/3$
  - From symmetry we infer $p=2/3$
- In addition to the two pure equilibria, we have one mixed equilibrium

Summary

- Individuals playing games
  - Two individuals moving simultaneously and playing rationally
- Strategies
  - Strictly dominant strategies (player is selfish!)
    - Two, one, or none
  - Nash equilibrium for pure strategies (both players assume other is selfish!)
    - One, more than one, or none
  - Nash equilibrium for mixed strategies (indifference, i.e., no advantage to pure)
    - Nash equilibria always exist, but might be mixed (they don’t always exist for pure strategies)

Part 2:
Extensions

Overview

- Pareto optimality and social optimality
  - Can we modify games to make them good for society?
  - Requires binding agreement among the players
  - This models regulation versus free market
- Dynamic games
  - Games played over time: sequence of moves observed by the players
  - Normal-form representation: choices represented simultaneously (table)
    - Optimize final result (precommit)
  - Extensive-form representation: choices represented sequentially (tree)
    - Optimize intermediate results (no precommit)

Pareto optimality and social optimality

Is a game “good for society”?

		Your partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Exam-or-presentation game: individual optimization is not globally best!
(exercise: is this a Nash equilibrium?)

- Remember the counterintuitive solution we found for this game
- In a Nash equilibrium, each player is optimizing individually
  - This does not mean that, as a group, the players are doing well
  - We saw examples with Prisoner’s Dilemma and exam-or-presentation
- We can change the game’s rules to get results that are “good for society”
  - Let’s make this idea precise!

Pareto optimality

- This definition originally comes from the Italian economist Vilfredo Pareto (1848-1923):
  - A choice of strategies (one by each player) is Pareto optimal if there is no other choice in which all players receive payoffs at least as high, and at least one player receives a **strictly** higher payoff
- This is an intuitive principle
  - Consider a choice that is not Pareto-optimal. Then there is an alternative choice where one player is better off without harming anyone.
  - This is clearly a superior alternative in any reasonable sense
  - If the players could jointly agree on this, then this is clearly a superior alternative ⇒ this is the problem! How can the players jointly agree?

Pareto versus Nash

- How does a Pareto optimal choice compare to a Nash equilibrium?
  - The Pareto-optimal choice is better, but it requires a binding agreement between them to actually play the superior pair of strategies
  - If the Pareto-optimal choice is not a Nash equilibrium, we have to disallow any player changing strategies to a better one (from the Nash viewpoint)

- Consider the exam-or-presentation game
  - Both studying for the exam is not Pareto optimal, because the choice of both preparing the presentation is strictly better for both
    - But without a binding agreement between the players, each player will still choose the exam because it increases their payoff
  - Both studying for the presentation is Pareto optimal
    - Note that the two outcomes where exactly one player prepares are also Pareto optimal!
    - Pareto optimal is a very strong condition because everyone has to do at least as well

Social optimality

- There is an even stronger condition:
  - A choice of strategies (one by each player) is socially optimal (or a social welfare maximizer) if it maximizes the sum of the players' payoffs
- In the exam-or-presentation game, the social optimum is achieved by both preparing for the presentation, which gives 90+90=180
- This definition only makes sense if it is actually reasonable to do arithmetic on the payoffs (adding them together)
  - It's not always clear that satisfactions can be added up like this
  - For example, one player can become rich and the other have a huge debt, as long as the sum is bigger it's ok
  - We can make intermediate conditions, where some people are allowed to become slightly poorer to make others better (tax the middle class but not the poor!)
- Socially optimal outcomes are always Pareto optimal (why?)

Free market versus regulation

- Nash equilibrium corresponds to free market
  - Payoffs are determined solely by individual choices
  - Completely decentralized, easy to implement
- Pareto optimality and social optimality correspond to regulation
  - Payoffs are determined by binding agreements between all players
  - Binding agreements require centralized negotiation (government)
- Human society requires a mix of both approaches
  - Pure free market can be disastrous for individuals
  - Too much regulation can be disastrous for companies
- Internet reflects human society, so it also has this trade-off
  - In this course we will see the trade-offs between the two approaches

Dynamic games

Dynamic games

- Dynamic games are played over time
    - A player moves, others observe the choice and then respond, and so on
    - So far we have focused on games played simultaneously

- Examples of dynamic games
    - Board games and card games where players alternate turns
    - Negotiations, with a sequence of offers and counteroffers
    - Bidding in an auction or pricing competing goods over time

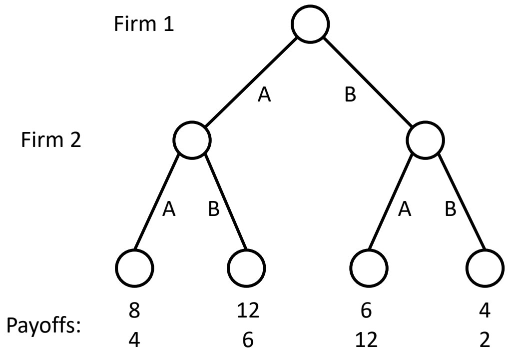
Normal and extensive forms of a game

- We need a new notation to specify a dynamic game
- Our previous notation is called normal-form representation
  - It specifies the list of players, their possible strategies, and the payoff matrix (payoffs by every possible simultaneous choice of strategies)
- We now define the extensive-form representation of a game
  - For dynamic games, this specifies the order of moves, what each player knows when they have the opportunity to move, what they can do when it is their turn to move, and what the payoffs are at the end

Example of a dynamic game

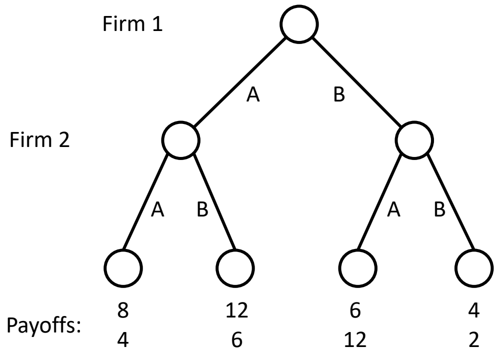
- We define an example game to illustrate the extensive form
  - Two firms, Firm 1 and Firm 2, try to decide whether to focus their advertising budget on two regions, A and B
    - Firm 1 chooses first
    - Firm 2 chooses second
  - How they divide up the profit
    - If Firm 2 follows Firm 1 into the same region, then Firm 1's "first-mover advantage" gives it 2/3 of the profit from that region, where Firm 2 only gets 1/3
    - If Firm 2 moves into the other region, then each firm gets all profit from its region
  - Total profit from each region
    - Region A has twice the market size of Region B, which means total profit from Region A is 12 whereas from Region B it is 6

Extensive-form representation: game tree



- We write the extensive-form representation as a game tree
- The tree is read from top to bottom
  - Top node represents Firm 1’s initial move
  - Two edges represent its choices
  - The terminal nodes represent the end of play and are labeled by payoffs
- A specific play corresponds to a path from the top node to a terminal node
  - Each player knows the complete history of past moves whenever making a new move

Reasoning in the extensive form



- We use the game tree to reason about the game
- If Firm 1 chooses A, how will Firm 2 choose?
  - Firm 2 maximizes payoff by choosing B
- Consider Firm 1’s opening move
  - If Firm 1 chooses A, it expects Firm 2 to choose B, giving payoff 12 for Firm 1
  - If Firm 1 chooses B, it expects Firm 2 to choose A, giving payoff 6 for Firm 1
  - Therefore, Firm 1 should choose A!
- This is a general approach
  - We start one step above the terminal nodes, where the last player has complete control
  - We then move up one level in the tree, and so on

Reasoning in the normal form (1)

Firm 2

		AA,AB	*AA,BB*	*BA,AB*	*BA,BB*
Firm 1	*A*	8, 4	8, 4	12, 6	12, 6
	B	6, 12	4, 2	6, 12	4, 2

- Another approach to analysis is to use the normal form
  - Each player makes a plan for how to play the entire game, covering all choices
  - This play corresponds to the player's strategy
- Firm 1 has two possible strategies, A or B
- Firm 2 has four possible strategies, which depend on what Firm 1 does
  - (A if A, A if B), (A if A, B if B), (B if A, A if B), (B if A, B if B) written as (AA,AB), …
  - We write it this way because Firm 2 has to enumerate all possibilities in advance!

Reasoning in the normal form (2)

Firm 2

		AA,AB	*AA,BB*	*BA,AB*	*BA,BB*
Firm 1	*A*	8, 4	8, 4	12, 6	12, 6
	B	6, 12	4, 2	6, 12	4, 2

- For Firm 1, A is a strictly dominant strategy
  - Firm 2 should play a best response to Firm 1: (BA,AB) or (BA,BB)
  - Therefore Firm 2 chooses B
- This gives the same prediction as the direct analysis of the game tree
  - From the game tree, we found also that Firm 1 chooses A and Firm 2 chooses B

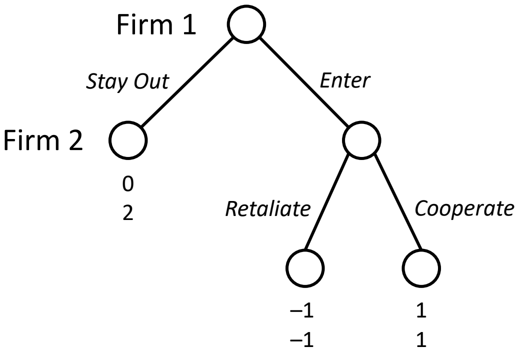
Extensive form versus normal form

Market Entry game (1)

- Dynamic games actually have two different models
  - It’s not always the same solution like in the previous example!
  - Let us see this by comparing the extensive form with the normal form

- Let’s present another game with our friends Firm 1 and Firm 2
  - Firm 2 is doing good business in a region
  - Firm 1 is considering whether to enter this market
- The first move is by Firm 1: enter or stay out
  - If Firm 1 stays out, the game ends with payouts (0,2)
- If Firm 1 enters, then the second move is by Firm 2
  - Firm 2 can cooperate or retaliate (price war), with payoffs (1,1) or (–1,–1)

Market Entry game (2)



Extensive-form representation

Firm 2

Firm 1	Retaliate	Cooperate
Stay Out	0, 2	0, 2
Enter	−1, −1	1, 1

Normal-form representation

- Analysis of extensive form:
  - At terminal node, Firm 2 chooses C
  - Therefore, Firm 1 will choose E
  - Payoff is (1, 1)
- Analysis of normal form:
  - Two Nash equilibria (E,C) and (S,R)
  - What does (S,R) mean?

Market Entry game (3)

- Extensive-form representation gives one solution (E,C)
- Normal-form representation gives two Nash equilibria
  - (E,C) corresponds to extensive-form analysis
  - (S,R) is unexpected: what does it mean?
- Normal form enumerates all possibilities in advance
  - (S,R) corresponds to Firm 2 committing in advance to a computer program that will automatically retaliate in case that Firm 1 enters the market
  - Firm 1 commits to a program that will stay out of the market
  - Given this pair of choices, neither firm has an incentive to change
- This shows the difference between the two forms!

Normal form versus extensive form

- In normal form, each player commits ahead of time to a complete plan for playing the whole game
- In extensive form, each player makes an optimal decision at each intermediate step
  - If Firm 2 truly commits in advance to retaliating, then (S,R) makes sense
  - But the dynamic game is not defined like this: Firm 2 only gets to evaluate its decision once Firm 1 has already entered the market
    - At that point, it is better to cooperate
- The standard model for dynamic games in extensive form assumes that players seek to maximize payoff at each intermediate stage
  - However, the other model can also be useful…

Different models for Market Entry

- Both models for dynamic games can be useful
  - Extensive form model: optimize at each intermediate step (no precommit)
  - Normal form model: optimize only the final result (by precommitting)
- The ability to commit to a particular course of action can be valuable
- If Firm 2 could make Firm 1 believe that it would retaliate in case of entry, then Firm 1 would choose to stay out
  - This gives higher payoff to Firm 2
- This suggests possibilities of action for Firm 2
  - For example, they could publicly announce that they will beat any competitor's price by 10% → this is a precommit
  - This can influence Firm 1's decision to enter or stay out

Summary

Summary

Individuals

- Strictly dominant strategies (selfish is best)
  - Two, one, or none
- Nash equilibrium for pure strategies (each assumes other is selfish)
  - One, more than one, or none
- Nash equilibrium for mixed strategies
  - Nash equilibria always exist, but might be mixed (not always exist for pure strategies)

Society

- Pareto optimality and social optimality
  - Add binding conditions so games are good for society instead of for individuals

Single move

Sequence of moves

- Dynamic games
  - Extensive form (intermediate situations are optimal; no precommits)
  - Normal form (only final result is optimal; assumes precommits)

LINFO1115
Reasoning about
a highly connected world

Lecture 5
Applications of game theory: traffic and auctions

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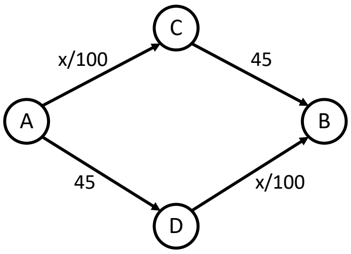
Applications of game theory

- We give two important applications of game theory
  - Both applications are useful for reasoning on networked systems
- Car traffic networks (chapter 8)
  - In a transportation network, players are the drivers, player strategies are the travel routes, and the payoff is the negative of the travel time
  - Braess's Paradox, adding roads can make travel slower, is explained with game theory
- Auctions (chapter 9)
  - In an auction, players are the bidders, player strategies are the amounts they bid, and the payoff is the "gain" if they win (how much they pay less w.r.t. the true value)
  - There are two main kinds of auctions: first-price sealed-bid auctions and second-price sealed-bid auctions
  - We analyze both kinds of auctions using game theory

Car traffic networks

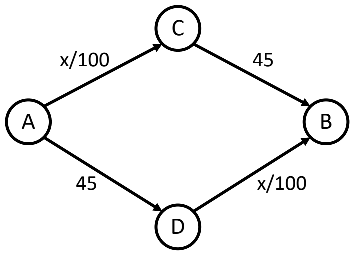
Chapter 8

Transportation network



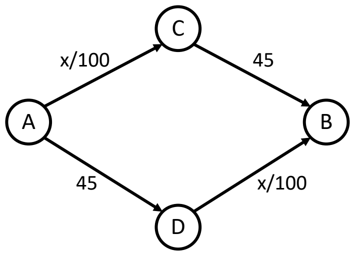
- We represent a transportation network by a directed graph
  - Each edge is labeled by the travel time in minutes when used by x cars
- We will see how this model responds to traffic congestion
- Assume 4000 cars want to get from A to B
  - If all cars take the upper route (through C), total time for every driver is 85 minutes
  - If the cars divide up the two routes evenly, total time for every driver is 65 minutes

Equilibrium traffic (1)



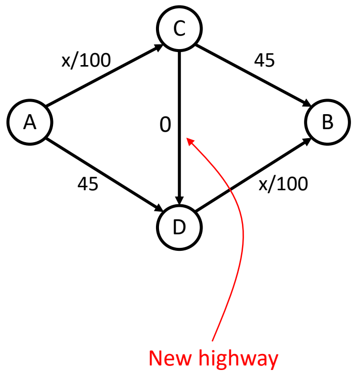
- This model is a game: players are the drivers
  - In general, each player's strategy is the possible routes from A to B
- In our example, each player has two strategies
  - Payoff is the negative of travel time (since we want to minimize the time)
- Compared to previous chapter, there are a huge number of players
  - But this is not a problem. Concepts of dominant strategies, mixed strategies, and Nash equilibria can be defined easily.

Equilibrium traffic (2)



- This game has Nash equilibria
  - Any list of strategies in which the drivers balance themselves equally between the two routes (2000 per route) is a Nash equilibrium
  - These are the only Nash equilibria (how many?)
- Why does equal balance give a Nash equilibrium?
  - With even balance, no driver has an incentive to switch to the other route
- Why do all Nash equilibria have equal balance?
  - Consider a list of strategies in which x drivers use the upper route and (4000-x) use the lower route
  - If x≠2000 then the routes have uneven travel times, and any driver on the slower route has incentive to switch

Braess’s Paradox



- In the previous example, self-interested behavior by all drivers causes them to balance perfectly between the available routes
- But with only a small change to the network, this is no longer true
  - The government decides to build a fast highway from C to D. For simplicity, assume its time is 0.
  - People’s travel time from A to B should get better, right?
- There is a unique Nash equilibrium in the new network, but with worse time for everybody
  - At equilibrium, every driver goes through both C and D
    - This gives 4000/100 + 0 + 4000/100 = 80 minutes
  - This is an equilibrium because no one benefits by switching
    - Every other route takes 85 minutes (verify this!)

Braess’s Paradox explanation

- The fast highway from C to D acts like a “vortex” that draws all drivers into it, to the detriment of all (because it is a dominant strategy)
- There is no way that self-interested behavior by drivers can get back to the even-balance solution that was better for everyone
- This phenomenon, that adding resources to a transportation network can hurt performance at equilibrium, was first postulated by Dietrich Braess in 1968
  - It has been observed empirically in real transportation networks, for example in Seoul, Korea, the destruction of a six-lane highway to build a public park for the Cheonggyecheon restoration project actually improved travel time into and out of the city (with the same traffic volume!)

Applications of Braess's Paradox

- The possibility to worsen network behavior by adding new routes occurs in many areas
- Electrical power grids (MPI for Dynamics and Self-Organization, 2012)
  - If power generation is decentralized, then adding additional links can reduce the transmission capability of the network as a whole
  - It is due to synchronization requirements: generators must have fixed phase relationships
  - If a new line is built to connect two generators, this changes their phase relationship, which can have an effect throughout the whole network
  - This problem needs to be recognized when transitioning to renewables, because they have a very large number of smaller generators
- Biological systems (Adilson E. Motter, 2011)
  - In management of endangered species food webs, the selective removal of a doomed species from the network can in some cases prevent further extinctions
- Sports teams (Skinner, Gastner, Jeong, 2009)
  - A basketball team is a network of possibilities to score a basket, with different efficiency for each pathway. Adding a star player could reduce the overall efficiency ot the team.
  - Soccer manager Helenio Herrera: "with 10 [players] our team plays better than with 11"

There is no paradox

- There are many situations where adding a new strategy to an existing game makes things worse for everybody
  - For example, consider the Prisoner's Dilemma: if the only strategy for each player were Not-Confess, then both players would be better off!
  - But the analogous phenomenon in a traffic network is more paradoxical
    - We have the intuition that "upgrading" a network must be good
- Much research has been done on game-theoretic study of traffic
  - Assume for number of cars x that travel time is a linear function ax+b
  - If we add edges to a network with an equilibrium, the new network has an equilibrium whose time is at most 4/3 as large (our example is almost this bad!)
    - Theoretical result of Tim Roughgarden and Éva Tardos
  - When edges can respond nonlinearly, it can get much worse

Auctions

Chapter 9

History of auctions

- Auctions have a long history, at least since 500 BC
  - In ancient Babylon, auctions of women were used for marriage
  - In ancient Rome, auctions were used to sell off spoils of a battle or to pay a debt
- In modern times
  - Since the 16th century, auctions of ships, artworks, tulip bulbs, goods, etc.
  - Christie's and Sotheby's use auctions to sell art
    - Yearly revenue in 2018 of $5 billion for Christie's and $4 billion for Sotheby's
  - Governments use them to sell frequency spectrum bands
- On the Internet
  - Online auctions (e-auctions) are common
    - They break down the physical limitations of traditional auctions, such as geography, presence, time, space, and the small size of the target audience
    - In 2002, online auctions accounted for 30% of all online e-commerce (mostly in search!)
  - The largest online auction site is eBay

Types of auctions (1)

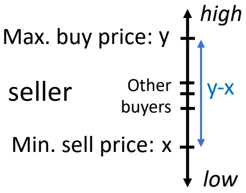
- We consider one seller auctioning one item to a set of buyers
  - The opposite situation, one buyer and a set of sellers, is called a procurement auction and is often used by governments to purchase goods
  - We will not talk specifically about procurement auctions
- When modeling auctions, the underlying assumption is that each bidder has an intrinsic value for the item being auctioned
  - We call this the bidder’s true value
  - The bidder is willing to pay any price up to this value, but no higher

Types of auctions (2)

- There are four main types of auctions (with many variations!):
- Ascending-bid auctions (English auctions) : interactive in real time
  - Seller gradually raises the price until only one bidder remains
- Descending-bid auctions (Dutch auctions) : interactive in real time
  - Seller gradually lowers the price until the first bidder accepts
- First-price sealed-bid auctions : simultaneous bids
  - The highest bidder wins and pays the value of their bid
- Second-price sealed-bid auctions (Vickrey auctions) : simultaneous bids
  - The highest bidder wins and pays the value of the second-highest bid
  - (Vickrey did game-theoretic analysis of auctions and won a Nobel Prize in 1996)

When are auctions appropriate?

- Auctions are not always appropriate
  - When you go to the supermarket, you do not bid for a bag of oranges
- Auctions are used when:
  - Buyers’ true values are unknown to others (unknown to seller and other buyers)
  - This is a common situation on the Internet, for example, for search advertising
- Known values ⇒ auctions are unnecessary
  - Consider where seller and buyers know each other’s values: seller values item at x and buyer values item at y
  - There is a surplus y-x that can be generated by the sale
  - If seller knows the true values and sells at y-ε, then full surplus goes to seller
    - Seller gets full surplus because he commits to a fixed price!
    - If buyer could commit, then some of the surplus would go to the buyer
- Unknown values ⇒ auctions are useful



Two kinds of unknown values

- First case: each buyer has an independent, private value
  - For example, each buyer is interested in the item for personal use
  - The values can be different because the buyers have different tastes
- Second case: there is a common value for all buyers
  - Assume each buyer plans to resell the item if bought
  - Then the item has an unknown but common value regardless of who buys it
- Auction strategies are different for private values and common values!
  - We will first study the case of independent, private values
  - After that, we will look at common values

Relationships between the auction formats

- Descending-bid auction is equivalent to first-price auction
  - In a descending-bid auction, buyers learn nothing while the auction runs
  - For each bidder i there is a first price b_i to buy
  - A descending-bid auction is equivalent to a sealed-bid first-price auction
    - The item goes to the bidder with the highest bid value, and this bidder pays that value

- Ascending-bid auction is equivalent to second-price auction
  - Consider an ascending-bid auction. How long should you stay in?
    - You should stay in until the price reaches your true value, and then drop out
  - The person with highest bid is the one who stays in the longest
    - This person pays the price at which the second-highest bidder dropped out!
  - The sealed-bid second-price auction is a simulation of an ascending-bid auction
    - It's a simulation because an ascending-bid auction has many bids in real time
    - (We will see later that bidding one's true value is a dominant strategy for both)

Second-price auction
in game theory

Second-price auction

- The sealed-bid second-price auction is widely used on the Internet
  - The auction format of eBay is essentially a second-price auction (why?)
  - The sponsored search markets used by search engines to sell keyword-based advertising use a generalization of the second-price auction
- Second-price auction theorem: Bidding your true value is a dominant strategy in a second-price sealed-bid auction
  - This is one of the most important results in auction theory
  - The best choice of bid is exactly what the object is worth to you!
  - We will formulate this result in game theory and prove it

We will see this later in Chapter 15

Example: sponsored search markets (1)

- Web search engines originally just found the most useful pages
  - But very soon it was realized that search can be combined with advertising to make money!
  - Keyword-based advertising, pioneered by Overture, is Google's main source of revenue
- Pay per click: the advertisers only pays when a user clicks on their ad
  - This increases the amount that advertisers are willing to pay
- Some examples
  - If you create an ad for "Keuka Lake" then users will go to your company's Web site when they click on it. This is a high value for the advertiser, and cost per click is around $1.50 (in 2010).
  - Top spot for query "calligraphy pens" costs $1.70 per click ("calligaphy pens" costs $0.60!)
  - For some queries, the cost is astronomical: for queries like "loan consolidation", "mortgage refinancing", "mesothelioma" the cost can be $50 per click!
    - If you query "mesothelioma" which is exposure to asbestos in the workplace, you are very likely suffering from it and want to sue your employer. The top queries are law firms!

Example: sponsored search markets (2)

- How are prices set for ads that appear with search queries?
  - Prices are set through an auction: it's too complicated to set prices manually!
- What auction is appropriate?
  - There are multiple slots and multiple advertisers allocated to them, it's not simple!
- Generalize second-price auction: the Vickrey-Clarke-Groves (VCG) Principle
  - Truthful bidding is a dominant strategy for VCG Principle, just like for second-price
    - This is important because it encourages truthful bidding and makes bidding easy
  - VCG Principle: "Each individual is charged a price equal to the harm he causes the other bidders by receiving the item"
    - This generalizes the second-price rule. If bidder 1 were not present, then bidder 2 who values the item at v_2 , would get the item. But since bidder 1 is present, then bidder 2 "loses" v_2 . The value v_2 is exactly what bidder 1 pays. The other bidders, 3 through n, are unaffected since they lose in both cases.

Second-price auction as a game

- We define the second-price auction in the language of game theory
  - Bidders i are the players
  - Bidder i's strategy is the amount b_i to bid as function of true value v_i
- Payoff to bidder i with value v_i and bid b_i is defined as follows:
  - If b_i is not the winning bid, then the payoff to i is 0. If b_i is the winning bid, and some other b_j is the second-place bid, then the payoff to i is $(v_i – b_j)$.

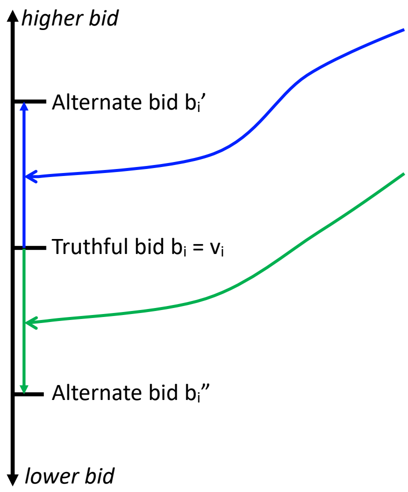
Note the j!

- Important detail: we need to handle the possibility of ties
  - What to do if two people submit the same bid and it is tied for largest?
  - We assume a fixed ordering on the bidders agreed to in advance
  - Winner in this case pays same as own true value, and has payoff of zero!

Truthful bidding in second-price auctions

- We claimed that your best bid is what the object is worth to you
- Second-price auction theorem:
  - In a sealed-bid second-price auction, it is a dominant strategy for each bidder i to choose a bid $b_i = v_i$
- Proof
  - To prove the theorem, we show that if bidder i bids $b_i = v_i$ then no deviation from this bid would improve the payoff, regardless of what strategy the other bidders use
  - There are two cases to consider: (1) i raises bid, and (2) i lowers bid
  - In both cases, the amount paid by the winner is determined entirely by the other bids, specifically by the largest among them (b_i does not determine payoff!)
  - When i changes their bid, all other bids remain the same, so payoff changes only if it changes i's win/loss outcome

Proof of second-price theorem



- Suppose bidder i chooses $b_i' > v_i$
  - Only affects payoff if i would lose with v_i and win with b_i'
  - Second-place bid b_j must be between b_i and b_i'
  - Payoff to i would be at most $v_i – b_j \leq 0$ (never positive!)
- Suppose bidder i chooses $b_i'' < v_i$
  - Only affects payoff if i would win with v_i and lose with b_i''
  - Before changing, v_i was winner and second-place bid b_k was between v_i and b_i''
  - Payoff before changing was $v_i – b_k \geq 0$ and after deviating it is 0 (since i loses)
- In both cases, i's payoff is not improved!
- Your bid determines whether you win or lose, but not how much you pay when you win

First-price auction
in game theory

First-price auction as a game

- First-price auctions are very different from second-price auctions!
  - In a sealed-bid second-price auction, the value of your bid determines whether you win or lose but not how much you pay
  - In a sealed-bid first-price auction, the value of your bid determines both whether you win or lose and how much you pay
- We define the first-price auction in the language of game theory
  - Bidders i are the players
  - Bidder i's strategy is the amount b_i to bid as function of true value v_i
- Payoff to bidder i with value v_i and bid b_i is defined as follows:
  - If b_i is not the winning bid, then the payoff to i is 0. If b_i is the winning bid then the payoff to i is $(v_i – b_i)$.

Note the i!

How to bid in a first-price auction

- Bidding your true value is no longer a dominant strategy
  - You would get payoff of 0 if you lose (of course!)
  - You would also get payoff of 0 if you win (since payoff is $v_i – b_i$)
- The optimal way to bid is to make your bid slightly less than v_i
  - Then you will get a positive payoff $b_i < v_i \Rightarrow (v_i – b_i) > 0$
- It is a trade-off between two opposing forces
  - If you bid too close to your true value, your payoff will be less
  - If you bid too low then you reduce your chance of winning
- It depends on your knowledge of the other bidders
  - For example, if there are many bidders, intuitively your bid should be higher (closer to your true value) because the highest competing bid will be higher

All-pay auctions

- An interesting variation of first-price auction is the all-pay auction
  - In an all-pay auction, all bidders always pay their bids!
- Payoff is defined as follows:
  - If b_i is not the winning bid, then the payoff to i is $–b_i$. If b_i is the winning bid then the payoff to i is $(v_i – b_i)$.

All losers pay!

- This kind of game happens often in the real world
  - Political lobbying: each lobbyist spends money, but only one will win
  - Design competitions: each design firm spends money, but only one will win
  - Research projects: each team spends time writing proposals, only one wins
- Analysis of all-pay auctions is similar to first-price auctions
  - The optimal bid is less than the true value, and there is a trade-off between bidding high (more chance of winning) and bidding low (increasing payoff)

Common values and the winner's curse

Common values

- In some cases, bidders have a common value for the object
    - In the two previous auctions, sealed-bid first-price and sealed-bid second-price, we assumed that bidders' true values are independent
    - But if the bidders intend to **resell** the object, then there is a common value (the price of the object when it is resold)
        - For example, bidders are antique stores that buy an antique object just to resell it
    - The common value is not always known: each bidder i has some information about the common value, giving an estimation v_i of the common value
- Common value model
    - We assume $v_i = v + x_i$ where v is the true common value and x_i is a random number with mean 0 that represents the error in i's estimate

Common values lead to winner’s curse

- Assume that an item with a common value is sold in a sealed-bid second-price auction
    - Is it still a dominant strategy for bidder i to bid v_i? No, it’s not, let’s see why.
- Assume that there are many bidders
    - The winning bidder will very likely overestimate the common value
        - What the winner pays, i.e., the second-place bid, will also likely be an overestimate
    - Therefore the winner will lose money on the resale of the object!
- This is called the winner’s curse
    - It is a general property of auctions where bidders want to resell the item
    - It was first studied in the petroleum industry, where firms bid on oil-drilling rights for tracts of land, and the common value is the value of the oil in the tract
    - It was also studied in baseball contract offers to baseball players, where the common value is the future performance of the player

Winner’s curse examples

- Suppose an oil field has a true intrinsic value of $10 million (amount of oil)
    - Oil companies guess the value with guesses from $5 million to $20 million
    - The company who wrongly estimated $20 million would win the auction
        - If it’s a second-price auction, they better hope the second price is less than $10 million!
- The winner’s curse occurs in many areas
    - Spectrum auctions where the government auctions frequency spectrum bands
    - IPOs (Initial Public Offering) where shares of a company are sold
        - Privately held company becomes a public company, “floating”, “going public”
    - Pay per click advertising
    - Retail supply (supermarkets): a supermarket chain asks for bids for its orange juice. You win the bid for a 2-year contract but you only have 3% profit margin. One month later, a hurricane hits orange-growing areas and price of oranges rises by 15%. You are now selling your orange juice at a loss!

Taking into account the winner’s curse

- Rational bidders will take the winner’s curse into account
- The bid should be a best estimate of the object’s value conditional on two things: (1) the private estimate v_i and (2) the fact of winning
  - It must be the case that winning is better than losing!
- This means that in a common-value auction, bidders will push their bids downward even when a second-price format is used
  - With a first-price format, bids will be pushed even further downward
- Determining the optimal bid is complex
  - We will not do it in this course

Summary

Summary

- Many real-world situations that don't look like games can be modeled using game theory
- Transportation networks (car traffic)
  - Adding roads increases drivers' strategies and can introduce worse equilibria!
  - Braess's Paradox is similar to the Prisoner's Dilemma
  - Many networks are subject to Braess's Paradox: power grids, biology, etc.
- Auctions
  - The most important kind of auction is the sealed-bid second-price auction
    - Bidding your true value is a dominant strategy (theorem and proof!)
  - We also look at other kinds of auctions, first-price and all-pay auctions
    - First-price: bid slightly less than your true value (depends on your knowledge) (why?)
    - All-pay: bid even less than first-price (why?)
  - We investigate common values and how they lead to the winner's curse
    - When done as second-price auction, bid less than true value (why?)
    - When done as first-price auction, bid further decreased (why?)

LINFO1115
Reasoning about a highly connected world

Lecture 6
Matching markets and market clearing

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Academic year 2022-23
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The course so far

The course so far…

- Introduction to graph theory (chapters 1-5) (Lectures 1 & 2)
  - Social networks with human beings
  - Formalizing human relationships: strong & weak links, positive & negative links
  - Graph evolution toward closure (three kinds of closure)
  - Graph evolution toward balance (strong and weak balance)
- Introduction to game theory (chapters 6, 8, 9) (Lectures 3 & 4)
  - Social networks with interactions between players
  - Formalizing interactions: players, games, payoffs, strategies (pure and mixed)
  - Individual rationality: Dominant strategies and Nash equilibria
  - Societal rationality: Pareto optimality and social optimality
  - Dynamic games: multiple moves played over time

Applications of game theory

Lecture 5

- Traffic networks (chapter 8)
  - Drivers are players, routes are strategies
- Auctions (chapter 9)
  - Bidders are players, bid amounts are strategies
  - Importance of the second-price auction

Lecture 6

- Matching markets (chapter 10) ← We are here
  - Buyers and sellers are players, their prices are strategies
  - Role of prices to regulate supply and demand

Lecture 7

- Markets with intermediaries (chapter 11)
  - Buyers, sellers, and traders are players, their prices are strategies
  - Equilibrium depends on network structure (monopoly versus competition)

Lecture 8

- Bargaining and power (chapter 12)
  - A node and its neighbors are players, the players negotiate (= multiple moves to find equilibrium)
  - Equilibrium depends on network structure (power of a node depends its place in the network)

Applications of graph theory

Lectures 9-13

- Information networks: using networks for sharing information
  - World-Wide Web (chapter 13)
  - Link analysis and Web search (chapter 14)
  - PageRank in depth (extra lecture)
- Cascades: when people influence other people
  - Information cascades (herding): sequential decision making (chapter 16)
  - Direct-benefit cascades: benefit from following the crowd (chapter 17)
  - Network cascades: diffusion on the network (chapter 19)
- Large-scale properties of social networks
  - Power laws and the long tail (chapter 18)
  - Small-world phenomenon (chapter 20)

Markets

Chapter 10-12

Importance of markets

- Networks are a platform where participants can interact
  - We saw one form of interaction, namely auctions
  - We now look at markets as another form of network-enabled interaction
- Markets are a general form of interaction between many participants
  - A market is any place where parties can meet to exchange goods or services, usually for money but sometimes directly through barter
  - A market is a uses prices as a coordinating mechanism to convey information among the parties to regulate production and consumption
  - Market economics can also be seen as an application of game theory
- Kinds of markets
  - Physical consumer markets, physical business markets
  - Non-physical markets including Internet markets
  - Financial markets for exchange of liquid assets (can be traded quickly)

How can we study market principles?

- Markets are complex and exist in many different variations
  - How can we study this in a formal way that helps us for networks?
  - In this lecture we introduce idealized market concepts
    - We will then explore these idealized concepts in a network context
  - In a real-world situation, we make the idealized concepts more realistic
    - The intuitions gained from the idealized concepts still hold
- We will study three kinds of idealized concepts

Today →

  - Matching markets: the right choice of prices can “match” buyers and sellers (ch. 10)
  - Markets with intermediaries: traders connect buyers and sellers in a network (ch. 11)
  - Bargaining and power: power to set prices depends on position in a network (ch. 12)

Recall: General approach of the course

- Real-world interactions of people on networks are complex
  - How can we understand them?
- We first study an idealized concept and then apply it to the real world
  - Idealized concept: simple formal model, properties, theorems
  - Real world: adapt intuitions of the idealized concept to the real world
- Examples we saw so far
  - Social-affiliation networks: formal model of how friendships are made
  - Structural balance: formal model of friends versus enemies
  - Games with rational players: formal model of competition and cooperation
  - Pareto and social optimality: formal model of benefits to society
  - Today → Matching markets: formal model of how prices regulate supply and demand

Matching markets

Chapter 10

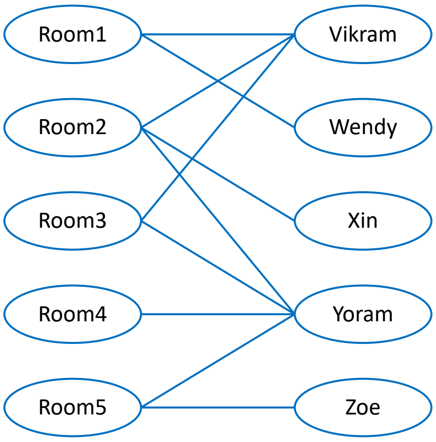
Matching markets

- Our first idealized concept is the bipartite matching market
  - A matching market is a market that matches each buyer to a desired item
  - Sometimes there is a perfect matching, a bijection between buyers and items
  - Perfect matchings are the goal, since they make everybody happy
  - Real-world markets will often converge toward perfect matchings
  - Prices are a decentralized mechanism to determine perfect matchings
    - We will prove that bipartite markets always have perfect matchings, "supply equals demand"

- Matching markets are an idealized way to study three principles
  - Prices do decentralized coordination of items to buyers
  - Prices can coordinate participants with different preferences
  - Prices can lead to socially optimal allocations

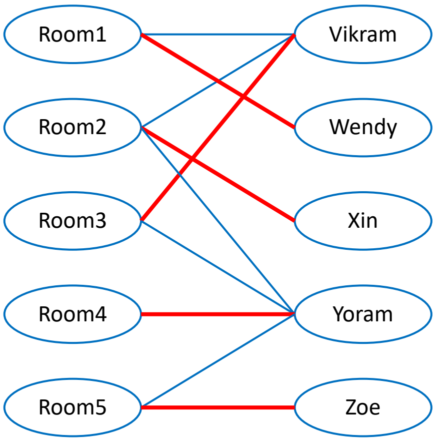
→ prices are a very powerful concept in a market economy!

A simple bipartite preference market



- Many matching markets are bipartite, where there are disjoint sets of buyers and items
- In scenario we define a simple preference market: a college assigns dormitory rooms to students
  - Each room accepts one student and each student lists several possible acceptable rooms
  - This example has five students and five rooms. Vikram lists Room1, Room2, and Room3 as acceptable, etc.
- Not all matching markets are bipartite
  - For example, Swaptree is a mail-based DVD swapping business where each customer gives a list of DVDs they want to own and a list they want to trade. Swaptree then finds sets of customers to exchange DVDs. In this market buyers and sellers coincide and there is no money.
  - We will however focus on bipartite markets!

A perfect matching



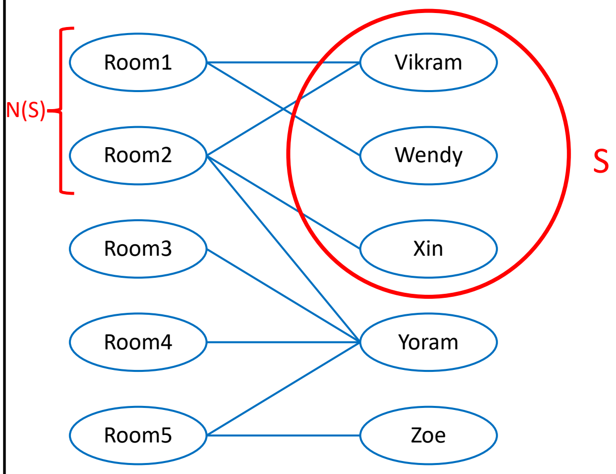
- The problem is to assign each student a room that they will be happy with
- The figure shows such an assignment
- This is called a perfect matching:

  If there are an equal number of nodes on each side of a bipartite graph, then a perfect matching is an assignment between sides such that:

  i. each node is connected by an edge

  ii. no two left nodes are assigned to the same right nodes

Constricted sets



- If a graph has a perfect matching, it is easy to demonstrate that fact
- But what if a graph has none?
  - How can you demonstrate this?
- There is a clean way to demonstrate this: the set {Vikram, Wendy, Xin} collectively has only two rooms that are acceptable
  - This set is called a constricted set
- Given any set S of nodes on the right side, define the neighbor set N(S) as the collection of all corresponding left nodes
- Define set S as *constricted* if |N(S)| < |S|

Matching theorem

- Matching theorem: If a bipartite graph with equal numbers of left and right nodes has no perfect matching, then it must contain a constricted set [Denes König 1931, Phillip Hall 1935]
- Proof approach (idea)
  - Given a bipartite graph with equal numbers of left and right nodes but no perfect matching
  - We consider a maximum matching, which has as many nodes as possible
  - We try to enlarge this matching by looking for a way to add one more node from each side
  - This search must fail, and when it does it has produced a constricted set

Matching market with prices

		Valuations (room 1, 2, 3)
Room1	Xin	12, 2, 4
Room2	Yoram	8, 7, 6
Room3	Zoe	7, 5, 2

- We now extend the simple bipartite preference market by adding prices
  - We add valuations (a form of price): each person gives a numerical value to each item
  - We define the quality of an assignment of persons to items as the sum of all persons' valuations
- An optimal assignment is an assignment of highest possible quality
  - {Xin-Room1, Yoram-Room3, Zoe-Room2} has quality 12+6+5 = 23
  - Prove it is optimal (exercise!)
  - Prove the original room assignment problem is a special case of the price model (exercise!)

Prices and market clearing

Housing market example

- We have a collection of buyers and a collection of sellers
  - Each seller i has a house for sale and each buyer wants to buy a house
  - Each buyer j has a valuation v_{ij} for each house held by seller i
  - Valuations are nonnegative integers, how much the buyer values the house
  - Sellers have valuation 0 for each buyer (they don't care who buys the house)
  - (This is just a different way of presenting room assignment with prices!)

- Each seller i puts his house up for sale at price $p_i \geq 0$
  - If buyer j buys the house, the buyer's payoff is $v_{ij} - p_i$
  - Buyer j wants to maximize payoff (selfish action, i.e., local action)
    - If there is a tie for maximum, any can be chosen (the preferred sellers of buyer j)
    - If $v_{ij}-p_i < 0$ for all sellers i, then buyer does not buy and payoff is 0

Prices can enable market-clearing

Calculate payoff for each buyer

- We have three sellers (a, b, and c) and three buyers (x, y, and z)
- Each buyer's valuations are listed on the right
- Each seller sets a price
  - Each buyer links to his preferred seller(s) (highest payoff)
  - Each buyer has bought a different house ⇒ the prices are market-clearing

The prices have resolved the contention for houses!

Prices do not always enable market clearing

Calculate payoff for each buyer

- We have three sellers (a, b, and c) and three buyers (x, y, and z)
- Each buyer's valuations are listed on the right
- Each seller sets a price
  - Each buyer links to his preferred seller(s)
  - Buyers x and z both value highest the house of seller a ⇒ the prices are NOT market-clearing

The prices have NOT resolved the contention for houses!

Prices enable market clearing (with ties)

Calculate payoff for each buyer

- We have three sellers (a, b, and c) and three buyers (x, y, and z)
- Each buyer's valuations are listed on the right
- Each seller sets a price
  - Each buyer links to his preferred seller(s)
  - Buyers y and z coordinate so each gets a different house ⇒ the prices are market-clearing
  - We eliminate the contention by taking advantage of multiple preferred sellers (ties)

The prices have resolved the contention for houses!

Market-clearing property

- In economics, market clearing means that the supply of an item is equal to the demand, so that there is no leftover of either
  - Economic theory often assumes that prices will adjust up or down to ensure market clearing: prices converge to "supply equals demand"
  - This is supposed to hold if all buyers and sellers have access to all information and that there is no "friction", i.e., impediments to price changes
- In bipartite matching markets, we say that a set of prices is market clearing if the resulting preferred-seller graph has a perfect matching
  - Each buyer ends up with a different house
  - Somehow, the prices have perfectly resolved the contention for houses!
    - This is despite the fact that all buyers value the house of seller a the highest

Two properties of market-clearing prices

- Market clearing prices might seem like magic: if prices are set the right way, then selfish buyers will automatically claim different houses
- In fact, we can always achieve “supply equals demand” :
  - In a bipartite matching market, there exists a set of market-clearing prices for any set of buyer valuations
  - (this is not totally obvious; we will prove it later on)
- In addition, market-clearing prices are always socially optimal:
  - For any set of market-clearing prices in a bipartite matching market, a perfect matching in the preferred-seller graph has the **maximum total payoff** for sellers and buyers, for any assignment of sellers to buyers
  - In other words, it maximizes the sum of all participants’ payoffs
- These two properties hold for bipartite matching markets
  - They are also true “in some form” for real-world markets → important intuition!

Proof of social optimality

- Social optimality can be proved with a simple argument
  - Consider M to be a perfect matching in the preferred-seller graph
- (1) Total payoff of sellers and buyers = total valuation of buyers
  - (Total payoff of sellers and buyers) = (Sum of all buyers $v_{ij}–p_i$) + (Sum of all sellers p_i)
  - Since each house gets bought, the $–p_i$ and $+p_i$ cancel
  - (Total payoff of sellers and buyers) = (Sum of v_{ij}) = (Total valuation of buyers)
- (2) Total valuation of buyers is maximized
  - (Total payoff of buyers) = (Sum of all buyers $v_{ij}–p_i$) = (Sum of v_{ij}) – (Sum of p_i)
  - (Total payoff of buyers) = (Total valuation of buyers) – (Sum of all prices)
  - But (Sum of all prices) is a constant
  - Each buyer maximizes its own payoff, so total valuation of buyers is also maximized!
- (3) Total payoff of sellers and buyers is maximized
  - This is exactly the definition of socially optimal!

Constructing a set of market-clearing prices

Why market-clearing prices always exist

- Let us try to understand why market-clearing prices always exist
- We will use a constructive argument:
  - We will take an arbitrary set of buyer valuations and give a procedure that determines seller prices and always arrives at market-clearing prices
  - The procedure is an auction: not a single-item auction as we saw before, but a more general auction that takes into account multiple items being auctioned and multiple buyers with different valuations [Demange, Gale, and Sotomayer 1986] [Egerváry 1916]
- We construct a procedure that gradually increases prices and we show that this will always lead to market clearing

Constructing market-clearing prices

- The process raises prices incrementally as follows:
  - Initially all sellers set their prices to 0
  - Buyers react by choosing their preferred seller(s)
  - If this graph has a perfect matching, we're done
  - Otherwise, there is a constricted set of buyers S
  - Consider the neighbors N(S), which is a set of sellers
  - Buyers in S only want what sellers in N(S) have, but there are fewer sellers |N(S)|<|S|
    - Sellers in N(S) are in "high demand", too many buyers are interested in them
    - So these sellers respond by simultaneously raising their prices by one unit, and we repeat
- In this way, the prices will gradually increase
  - We need to show that this will always lead to a perfect matching
  - Increasing prices in a bipartite matching market can always lead to market clearing

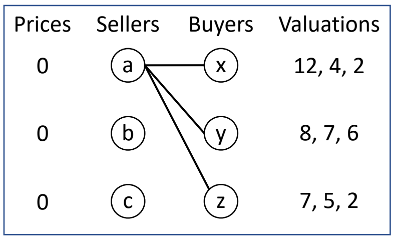
One round of the auction

i. At the start of the round, there is a current set of prices with the smallest one equal to 0
ii. We construct the preferred-seller graph and check whether there is a perfect matching
iii. If there is, we're done: the current prices are market-clearing → **Done**
iv. If not, we find a constricted set of buyers S and neighbors N(S)
v. Each seller in N(S) simultaneously raises his price by one unit
vi. If necessary, reduce the prices: subtract the same amount from each price so that the smallest price becomes zero (reduction step)
vii. We now begin the next round of the auction with these new prices

Loop

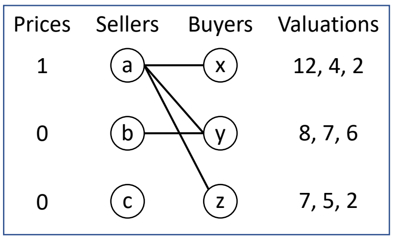
Example auction: round 1

Start of round 1



Prices	Sellers	Buyers	Valuations
0	a	x	12, 4, 2
0	b	y	8, 7, 6
0	c	z	7, 5, 2

End of round 1

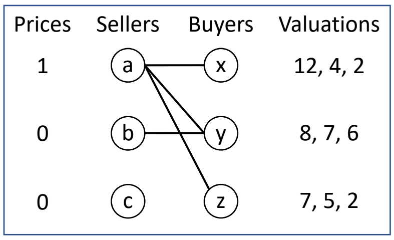


Prices	Sellers	Buyers	Valuations
1	a	x	12, 4, 2
0	b	y	8, 7, 6
0	c	z	7, 5, 2

- In round one, S={x,y,z} and N(S)={a}, so a increases its price by one
- The auction continues to the second round

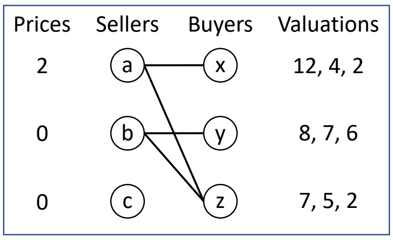
Example auction: round 2

Start of round 2



Prices	Sellers	Buyers	Valuations
1	a	x	12, 4, 2
0	b	y	8, 7, 6
0	c	z	7, 5, 2

End of round 2

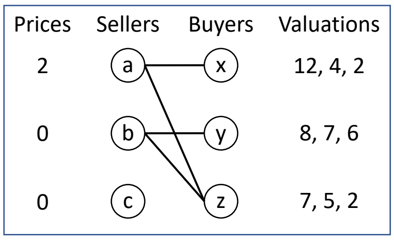


Prices	Sellers	Buyers	Valuations
2	a	x	12, 4, 2
0	b	y	8, 7, 6
0	c	z	7, 5, 2

- In round two, S={x,z} and N(S)={a}, so a increases its price by one
- The auction continues to the third round

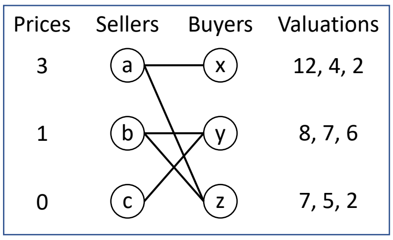
Example auction: round 3

Start of round 3



Prices	Sellers	Buyers	Valuations
2	a	x	12, 4, 2
0	b	y	8, 7, 6
0	c	z	7, 5, 2

End of round 3

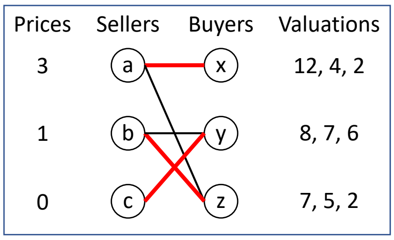


Prices	Sellers	Buyers	Valuations
3	a	x	12, 4, 2
1	b	y	8, 7, 6
0	c	z	7, 5, 2

- In round three, S={x,y,z} and N(S)={a,b}, so a and b both increase their prices by one
- The auction continues to the fourth round

Example auction: round 4

Start of round 4



Prices	Sellers	Buyers	Valuations
3	a	x	12, 4, 2
1	b	y	8, 7, 6
0	c	z	7, 5, 2

- In round four, there is a perfect matching!
- The auction ends and the prices are now market clearing
  - Note that other market-clearing prices are possible

Proving that the auction will always end

- The auction can only end if it reaches a set of market-clearing prices
  - If we show the auction always ends, then market-clearing prices always exist
- How can we show this in general?
  - We will define a nonnegative "potential energy" that always decreases
  - Potential of a buyer = maximum payoff it can currently get from any seller
  - Potential of a seller = current price it is charging
  - Potential energy of the auction = sum potential energy of buyers and sellers
- Evolution of the potential energy
  - Initially, all sellers have potential 0 and each buyer has maximum valuation
  - So the initial potential energy is some integer $P_0 \geq 0$

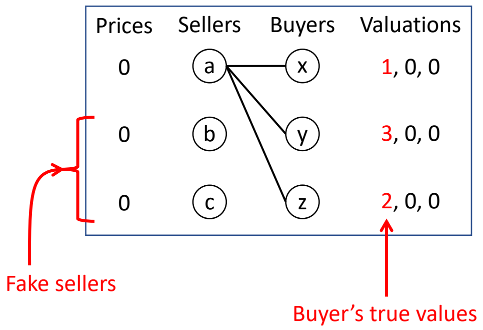
Evolution of the potential energy

- Potential energy only changes when the prices change at (v) and (vi)
- Reduction step (vi): subtract p from each price
  - Each seller potential goes down by p
  - Each buyer potential goes up by p

  } No effect on potential energy of auction
- Step (v): all sellers in N(S) raise prices by 1
  - Each seller potential in N(S) goes up by 1
  - Each buyer potential in S goes down by 1
  - Since $|S|>|N(S)|$, the potential energy of the auction decreases by at least 1
- The potential energy of the auction starts at $P_0 \geq 0$ and cannot drop below 0, so the auction must come to an end within P_0 steps

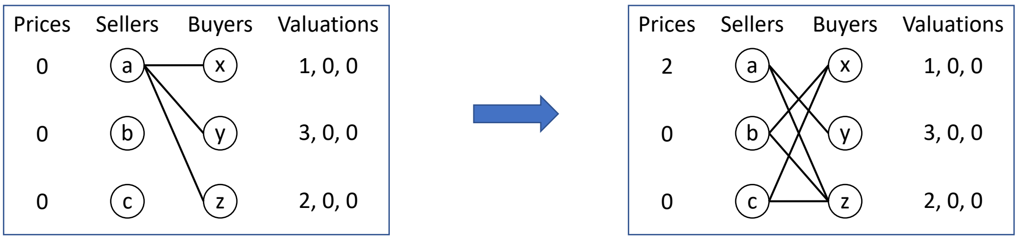
Connection to single-item auctions

Mapping the single-item auction



- The bipartite matching market we have just seen is a kind of complex auction
  - Sellers keep increasing prices until there is market clearing, where all buyers have bought an item
- How does this relate to the single-item auctions we saw before?
  - The single-item auction is a special case of the bipartite graph auction!
- Assume we have n buyers and a single seller auctioning an item
  - Let buyer j have valuation v_j for the item
  - We need equal numbers of buyers and sellers, so we add n-1 “fake” sellers
  - We give buyer j a valuation 0 for the fake sellers

It is exactly an ascending-bid auction



Buyer y wins and pays 2

- The price-raising procedure corresponds to raising prices in a single-item auction
  - Initially all buyers identify the real seller, constricted set S = all buyers, and N(S) = the single real seller
- The real seller raises price by one unit
  - This continues as long as at least two buyers have the real seller as their unique preferred seller
- Finally, only one buyer is left who prefers the real seller
  - This is exactly when the buyer with second-highest valuation drops out, so **the price is equal to second-highest valuation**
  - This implements exactly an ascending-bid auction (English auction)

Auctions and markets are related

- The bipartite matching market generalizes an ascending-bid auction
  - The ascending-bid auction has n buyers and 1 seller. The seller keeps increasing prices until one buyer wins the auction.
  - The bipartite matching market has n buyers and n sellers. The sellers keep increasing prices until each buyer has found one seller.
- There is a deep connection between auctions and markets
  - The bipartite matching market is a form of second-price auction with n sellers instead of 1 seller, and where all buyers “win” an item
  - We have shown this connection for the bipartite matching market (our idealized model), but it is also true in some form for the real world

Summary

Summary of matching markets

- Markets are a general form of interaction between participants
  - Markets use prices as a decentralized coordination mechanism
- We introduce an idealized market and study its behavior
  - A bipartite matching market with same numbers of buyers and sellers
  - Market clearing happens when all buyers are connected to different sellers
    - This corresponds to a perfect matching in our bipartite matching market
  - We show that prices can always be found to give market clearing
    - Furthermore, we show that market-clearing prices are always socially optimal
    - This gives an idealized view of how markets regulate prices
    - It is actually a kind of auction, generalizing the ascending-bid (English) auction
- In real-world markets, it is often assumed that “supply equals demand”
  - Market clearing makes precise the idea of “supply equals demand”
  - Prices are always converging to values that cause “supply equals demand”
  - The idealized market we saw can be adapted to real-world situations

LINFO1115
Reasoning about a highly connected world

Lecture 7
Markets with intermediaries

Peter Van Roy

Academic year 2021-22
École Polytechnique de Louvain
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Our study of markets continues!

- Markets are a general form of interaction between participants
  - All participants win from being part of the market (unlike auctions where only one wins)
  - The study of markets is the core of economics
  - To understand markets, we will study three idealized concepts
    - The intuitions we gain from the idealized concepts can be applied to real-world markets!
- Matching markets: prices are a decentralized coordination mechanism (ch. 10)
  - The right choice of prices will match buyers and sellers ("supply equals demand")
  - Prices naturally converge toward a matching
  - In a perfect matching, each buyer maximizes its own payoff and the total payoff is socially optimal
- Today Markets with intermediaries (ch. 11)
  - Traders connect buyers and sellers in a network, creating monopolies and competition
- Bargaining and power in markets (ch. 12)
  - Position in the network is important for negotiations between nodes

Markets with intermediaries

Chapter 11

Markets with intermediaries

- Last week we studied a market where buyers and sellers interact directly
  - A bipartite matching market with same numbers of buyers and sellers
  - We saw that prices can always be found to give market clearing
  - This shows how prices are a decentralized coordination mechanism
- But buyers and sellers often interact through intermediaries
  - Each buyer and seller knows a small number of intermediaries
  - This simplifies transactions for the buyers and sellers
    - A network with traders only has O(n) links instead of O(n^2) links for direct interaction
  - But it comes at the cost of giving intermediaries some power
- Today we will study markets with intermediaries
  - We show how to compute their power which depends on the network's structure

Introductory example: the stock market

Stock market

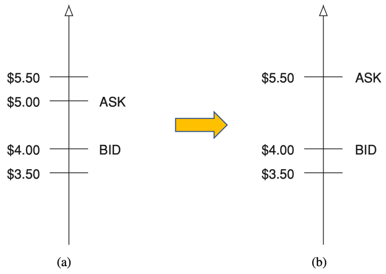
- A stock market is an intermediary in buying and selling shares; there are many variations!
  - Different ways of setting prices
    - Market-clearing (like chapter 10) or just matching orders with prices set elsewhere
    - Prices set by specialists or by algorithms
  - Different schedules: continuously during the day or with batches
  - Different participants: anyone may access or only restricted groups may access
- Stock exchanges in USA
  - New York Stock Exchange (NYSE)
  - NASDAQ-OMX
  - Alternative trading systems: Direct Edge, Goldman Sachs, Investment Technologies Group (ITG)
- Stock exchanges in Europe
  - Euronext: Pan-European Dutch-domiciled France-headquartered
  - London Stock Exchange Group
  - Deutsche Börse
  - SIX Swiss Exchange
  - Nasdaq Nordic

Stock markets and order books

Pour les francophones:
Ask: “je demande de l’argent pour vendre”
Bid: “j’offre de l’argent pour acheter”

Ask: desire to sell for minimum $5.00

Bid: desire to buy for maximum $4.00



Trader has bought shares at $5.00 (market order)

- For each stock that is traded, the market creates an order book
  - An order book is a list of orders that buyers and sellers have submitted
  - Lowest offer to sell is called the ask
  - Highest offer to buy is called the bid
- The order book is managed by a specialist / trader (works at the stock market)
  - Traders make offers to buy and sell, for example at market prices (market order)
  - If you want to buy or sell stock, contact a trader (talk to your bank)!

Complexification of the stock market

- Actual structure of the stock market is complex and evolving
- Large traders split their orders into many pieces over the day
  - Mutual funds, banks, pension funds, insurance companies, hedge funds
  - They submit the pieces to many different trading systems
  - They want to hide their trading desires, since that would influence the market
- Dark pools (Sigma-X by Goldman Sachs, ITG)
  - Private exchanges that are not accessible by the investing public
  - Created to allow institutional investors to avoid impacting the market
  - 40% of all US stock trades in 2017, 16% in 2010

A model of
trade on networks

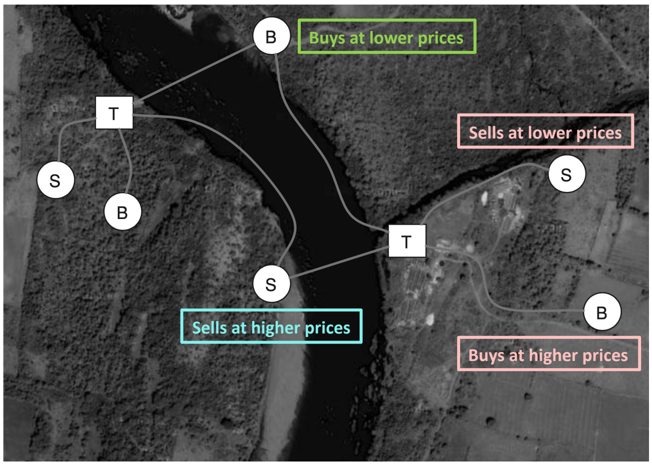
From one market to a network of markets

- Many buyers and sellers
  - Previously we had one market directly connecting buyers and sellers
    - Our idealized “bipartite matching market”
  - In real life there are many markets connecting buyers and sellers
- Many intermediaries
  - In real life, buyers and sellers are often not connected directly
- Network structure
  - Buyers and sellers are connected to intermediaries (traders, stores)
  - Now we have a network of buyers, sellers, and traders
- Main question: what is the equilibrium in this network?
  - How can we reason about this? Again, let us define an idealized model!

Buyers, sellers, and traders

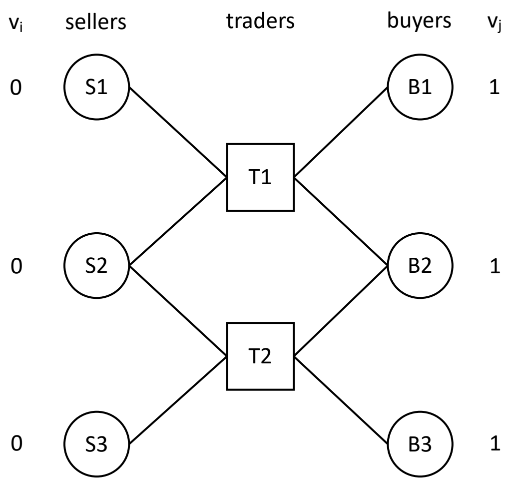
- We will define an idealized model with buyers, sellers, and traders
- Three fundamental principles
  1. Individual buyers and sellers trade through intermediaries
  2. Buyers and sellers may have access to different intermediaries
  3. Buyers and sellers may trade at different prices
- Prices depend on the positions in the network
  - If a buyer can only access one trader, then the trader increases his payoff
  - If a buyer can access several traders, then the buyer increases his payoff
- Depending on the positions, we will compute the equilibrium prices

Agricultural trade network



- Sellers S, buyers B, traders T
  - Edge shows when nodes can trade
- S and B on the right only access T on their side of the river
  - This S will sell at lower prices (**bad** for S)
    This B will buy at higher prices (**bad** for B)
- B at the top accesses two T nodes
  - This B will buy at lower prices (**good**!)
- S on the bottom accesses two T's
  - This S will sell at higher prices (**good**!)
- We will make these intuitions precise!

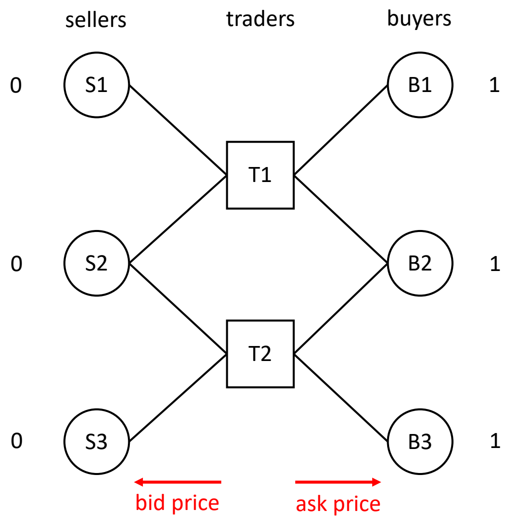
Graph model of this trading network



- Each seller i holds one unit of the good, which he values at v_i
  - He can sell it for a price ≥ v_i
- Each buyer j values one unit of the good at v_j
  - He can buy it for a price ≤ v_j
  - He only wants one unit of the good
- Everyone knows these valuations
  - This model describes interaction between individuals who know each other (they have a history of trade with each other)
- Simplifications w.r.t. matching markets
  - All buyers have the same valuation
  - Network is fixed and externally imposed

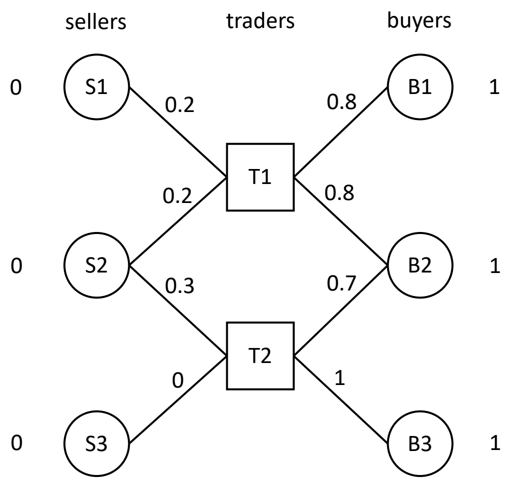
Added to simplify our analysis

Prices and the flow of goods (1)



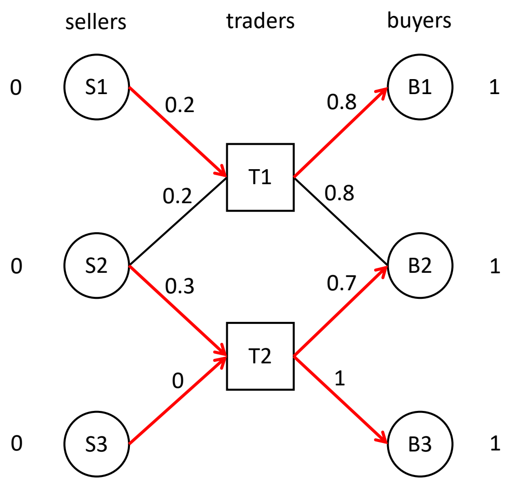
- Flow of goods from sellers to buyers is determined by a game
  - First, traders set prices
  - Second, sellers and buyers react
- First step:
  - Each trader t offers a bid price b_{ti} to each seller i he is connected to
    - An offer by t to buy i's good at price b_{ti}
  - Each trader t offers an ask price a_{tj} to each buyer j he is connected to
    - An offer by t to sell j's good at price a_{tj}

Prices and the flow of goods (2)



- Second step:
  - Each seller and buyer chooses at most one trader to deal with
  - Each seller sells a copy (or not)
  - Each buyer buys a copy (or not)
- At most one copy of the good moves along any edge in the network
  - Many goods can pass through one trader
  - A trader sells exactly what he receives; there are penalties if he defaults or has excess inventory (neither will happen)

Prices and the flow of goods (3)



- This determines the flow of goods
  - From each seller, through traders, to each buyer
  - Red arrows show how goods flow
- Sometimes prices are pushed to the limit
  - S3 accepts the bid even though it is equal to his value (zero payoff)
  - B3 accepts the ask even though it is equal to his value (zero payoff)
  - In fact, S3 and B3 are indifferent between accepting and rejecting the offer
  - We (the modelers) handle indifference by choosing either alternative
  - This models what happens when prices are pushed to the limit

Key principle: indifference

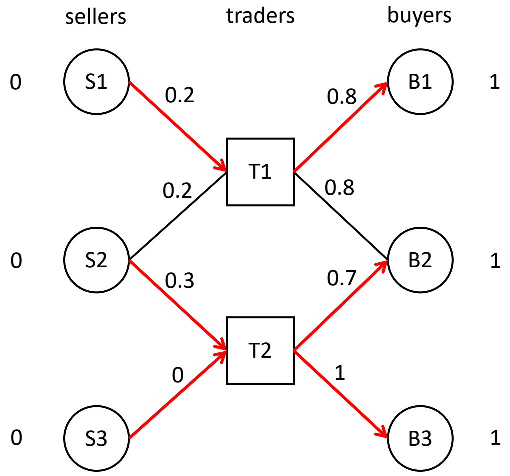
- Many transactions take place at an edge value
  - Prices are pushed to the limit of an individual's willingness to trade
  - We could assume that a minuscule positive payoff (one cent, 0.01) is required for a trade to happen, so the bid and ask values would be 0.01 and 0.99
  - This is actually messier to reason about, so we do not do this

- We allow trades with zero payoff with ties broken as needed
  - It's just a formal way to represent a price or payoff being driven almost to 0
  - You can imagine that the prices are shifted by very small amounts like 0.01 to account for the decision that is taken
  - This technique is called indifference: it allows the network modelers to break ties as required

Strategies and payoffs

- Strategies
  - A trader's strategy is to choose the bid and ask prices
  - A seller or buyer's strategy is a choice of neighboring trader (or not)
- Payoffs
  - A trader's payoff is his total profit: sum of all ask prices of accepted offers to buyers minus sum of bid prices of accepted offers to sellers
  - For seller i, payoff from selecting trader t is b_{ti} while payoff from selecting no trader is 0 (note we only consider when $v_i = 0$)
  - For buyer j, payoff from selecting trader t is $v_j – a_{tj}$ while payoff from selecting no trader is 0. In the former case, buyer receives the good and pays a_{tj} .

Prices and the flow of goods (4)



Payoffs for this example:

- T1’s payoff is (0.8 – 0.2) = 0.6
- T2’s payoff is (0.7 + 1) – (0.3 + 0) = 1.4

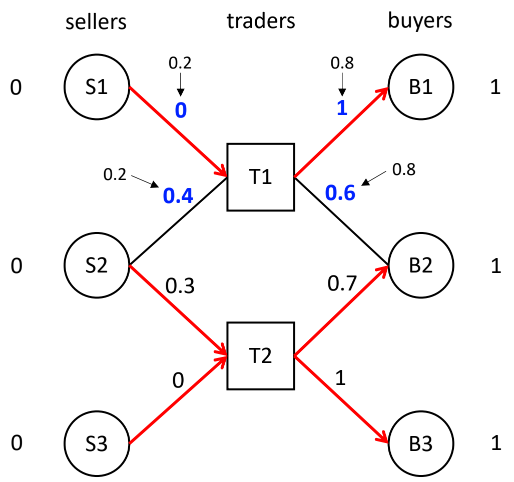
- S1’s payoff is 0.2
- S2’s payoff is 0.3
- S3’s payoff is 0

- B1’s payoff is (1 – 0.8) = 0.2
- B2’s payoff is (1 – 0.7) = 0.3
- B3’s payoff is (1 – 1) = 0

Two-stage structure of the game

- The game we define here has two stages
  - In our previous games, all players moved simultaneously
- First stage: all traders simultaneously choose ask and bid prices
  - This is where the strategic reasoning is done
- Second stage: all sellers and buyers simultaneously choose traders
  - The second stage is extremely simple, each seller and buyer just chooses the trader with the best offer
- When we discuss equilibria, we need to consider both stages however
  - There is a significance to having two stages

Reasoning about strategies



- Let’s consider the strategies of T1 and T2
- Trader T1 is making bad decisions!
  - If he were to raise his bid to S2 to 0.4 and lower his ask to B2 to 0.6, he would steal the trade from T2: both S2 and B2 would choose him and he would gain a profit of 0.2
  - He could lower his bid to S1 and raise his ask to B1, because even with worse offers S1 and B1 are still forced to deal with him.
- The blue shows what T1 could do
  - His payoff has increased to (1 + 0.6) – (0 + 0.4) = 1.2
  - S1 and B1 are now indifferent, we (the modelers) decide what they do!
- But it’s not over!
  - Let’s define an equilibrium concept for this game

Equilibria in trading networks

Generalizing the Nash equilibrium

- Let’s consider carefully how to choose strategies…
  - It is still each player choosing a best response to the other, like for Nash equilibrium
- In the first stage, traders choose a best response to what the sellers and buyers will do, and to what the other traders do
- In the second stage, sellers and buyers choose a best response to whatever the other players are doing
- The difference is that the sellers and buyers move second
  - Sellers and buyers must choose optimally whatever prices the traders choose, and the traders know this!
  - This is called “subgame perfect Nash equilibrium”, we’ll just call it “equilibrium”

Equilibrium in trading networks

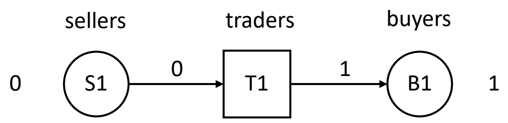
- We reason as follows about equilibria in the trading network:
  - We mainly think about the strategies of the traders in the first stage
    - Just as in the simultaneous move game we saw before
  - Sellers and buyers then choose best offers in the second stage
    - Their best responses are simple, so this is easy to take into account

- We can now work out the possible equilibria for trading networks

  There are two possible building blocks (two “patterns”):

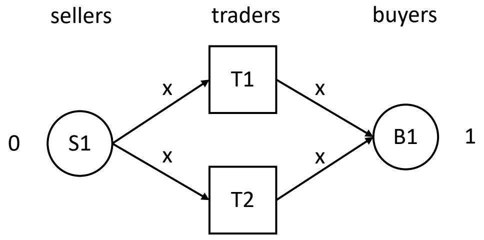
  1. Buyers and sellers who are monopolized (only a single trader)
  2. Buyers and sellers who benefit from perfect competition (multiple traders)

Monopoly



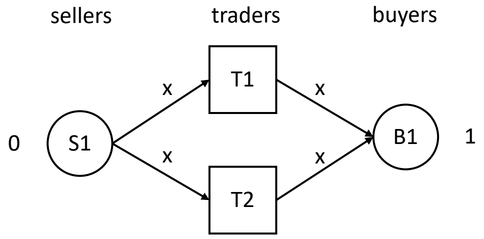
- Buyers and sellers are subject to monopoly when they access only a single trader
- The only equilibrium is for the trader to set bid 0 to the seller and ask 1 to the buyer
  - Seller and buyer will accept these prices; the good will flow from S1 to T1 to B1
- We can prove this is an equilibrium
  - For any other bid and ask between 0 and 1, the trader could slightly lower the bid or raise the ask, thereby achieving higher profit

Perfect competition (1)



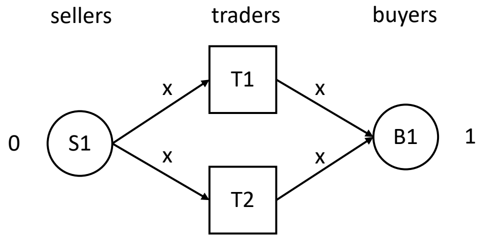
- Buyers and sellers are subject to perfect competition when they access more than one trader
- There is competition between traders T1 and T2 to buy the copy of the good from S1 and sell it to B1
- Suppose T1 is doing the trade with bid b to the seller and ask a to buyer with a>b
  - T2 has payoff of zero, but that is not a best response. T2 could offer bid slightly above b and offer ask slightly below a, and take away the trade from T1, with payoff>0
  - So the trader T1 performing the trade must have a payoff of 0, with same x for bid and ask. This equilibrium involves indifference on his part, between trading and not trading.

Perfect competition (2)



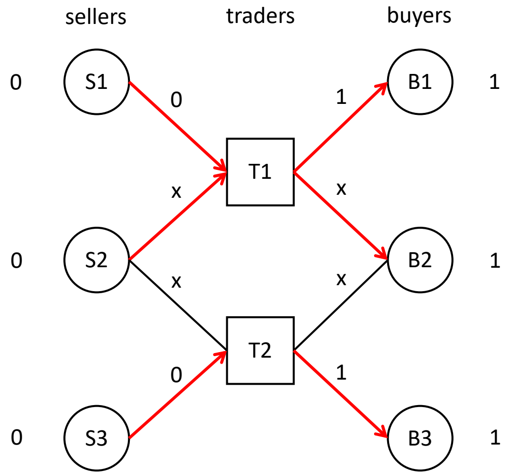
- Suppose T2 is not doing the trade
  - T2 must be offering a bid b≤x or else the seller would sell to T2
  - T2 must be offering an ask a≥x or else the buyer would buy from T2
  - Furthermore, if a>b then T1 could raise bid b or lower ask a to perform the trade while still making a profit
- Therefore, equilibrium for T2 occurs at common bid and ask of x

Perfect competition (3)



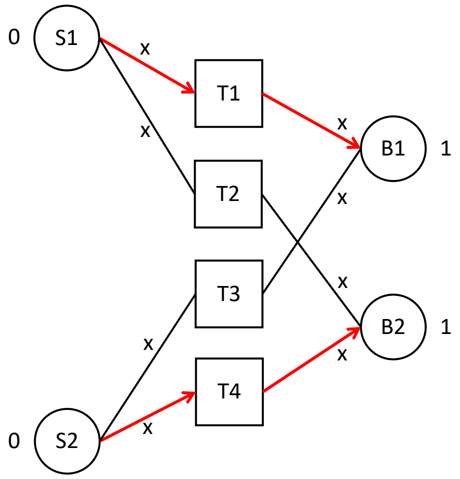
- Equilibrium for both T1 and T2 occurs at common bid and ask of x
- What is the value of x? From our model, all we know is 0≤x≤1.
- Value of x determines the profit of seller and buyer (x and 1-x, respectively)
  - The choice of x reflects something about the relative power of seller and buyer
  - We need to look outside the formulation of the trading game to determine x

Equilibria for the agricultural network



- We use the monopoly and perfect competition building blocks to work out the equilibria for our example network
- Sellers S1 and S3, buyers B1 and B3 are monopolized, driving prices to 0 and 1
- Seller S2 and buyer B2 benefit from perfect competition
- What are the payoffs for buyers and sellers for these equilibria? [exercise]
- What is the value of x?

Implicit perfect competition



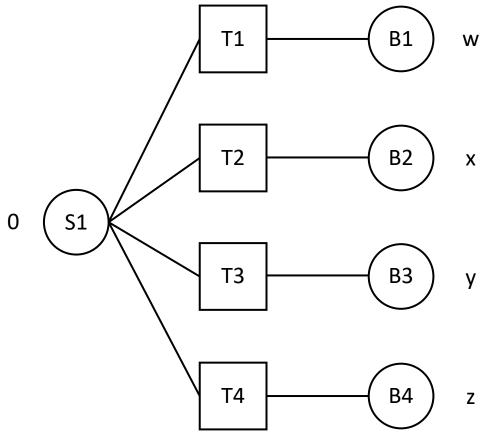
- In our previous examples, when a trader makes no profit it is because there is another trader who is connected to the same seller and buyer
- Traders can make zero profit for other reasons, based on the global structure of the network
  - Here is an example!
- It is easy to see this is an equilibrium: we simply check that each trader has a best response to all other traders [exercise]
- It is harder to verify that all bid and ask prices are the same value x
  - Check alternatives where some trader has bid<ask and identify a deviation [exercise]

Other equilibrium phenomena

Other equilibrium phenomena

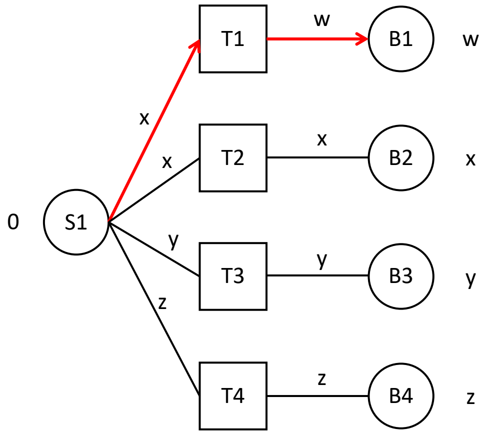
- Our network model is expressive enough to represent more phenomena than just monopoly and perfect competition
  - This shows the usefulness of the idealization
- We give two examples:
  - Second-price auctions: we show how they can arise from a trading network
  - Ripple effects: we show how small changes to the network structure can affect payoffs of nodes not directly involved in the change

Second-price auction (1)

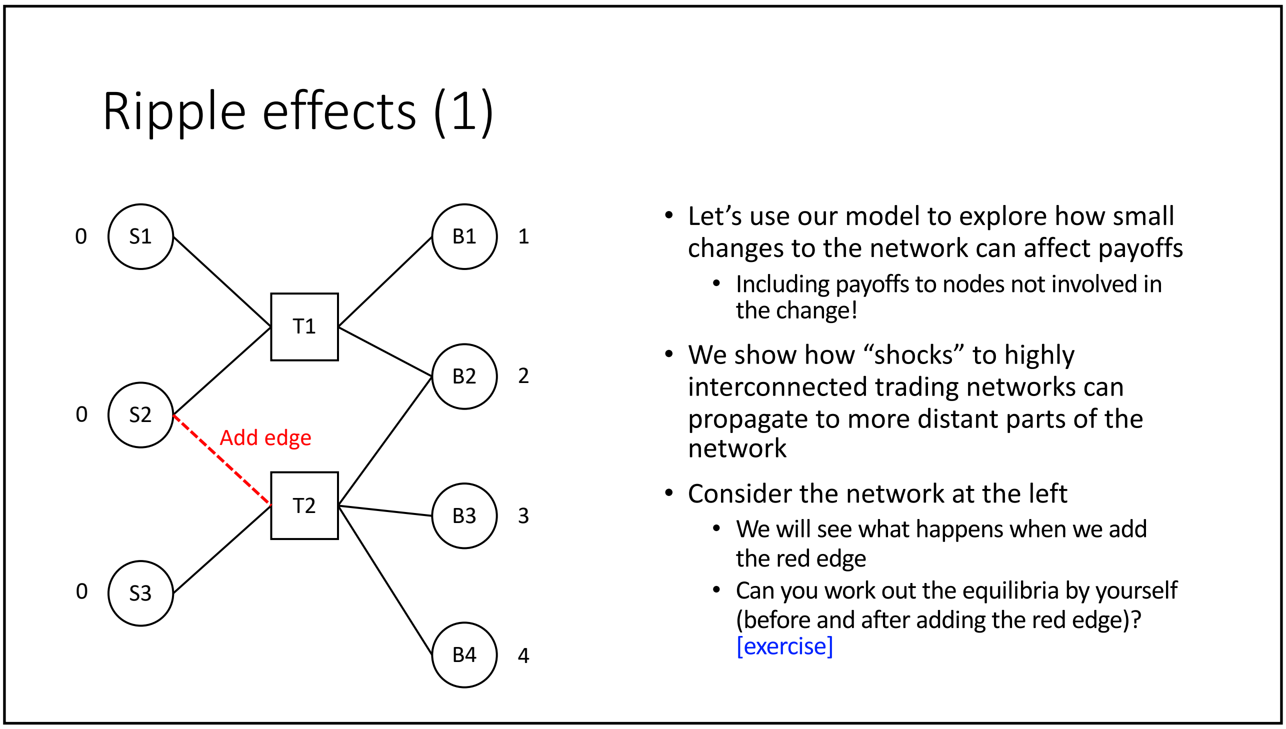


- We can present the structure of a single-item auction using a trading network
- Four potential buyers value the item at w > x > y > z (ordered for simplicity)
  - Each buyer has one trader, as a “proxy”
- What is the equilibrium?
  - T1 can outbid all other traders, since he can buy from the seller for value w
  - T1 will outbid the others by the minimum needed to make the trade
  - So what will happen? [do exercise before checking next slide]

Second-price auction (2)

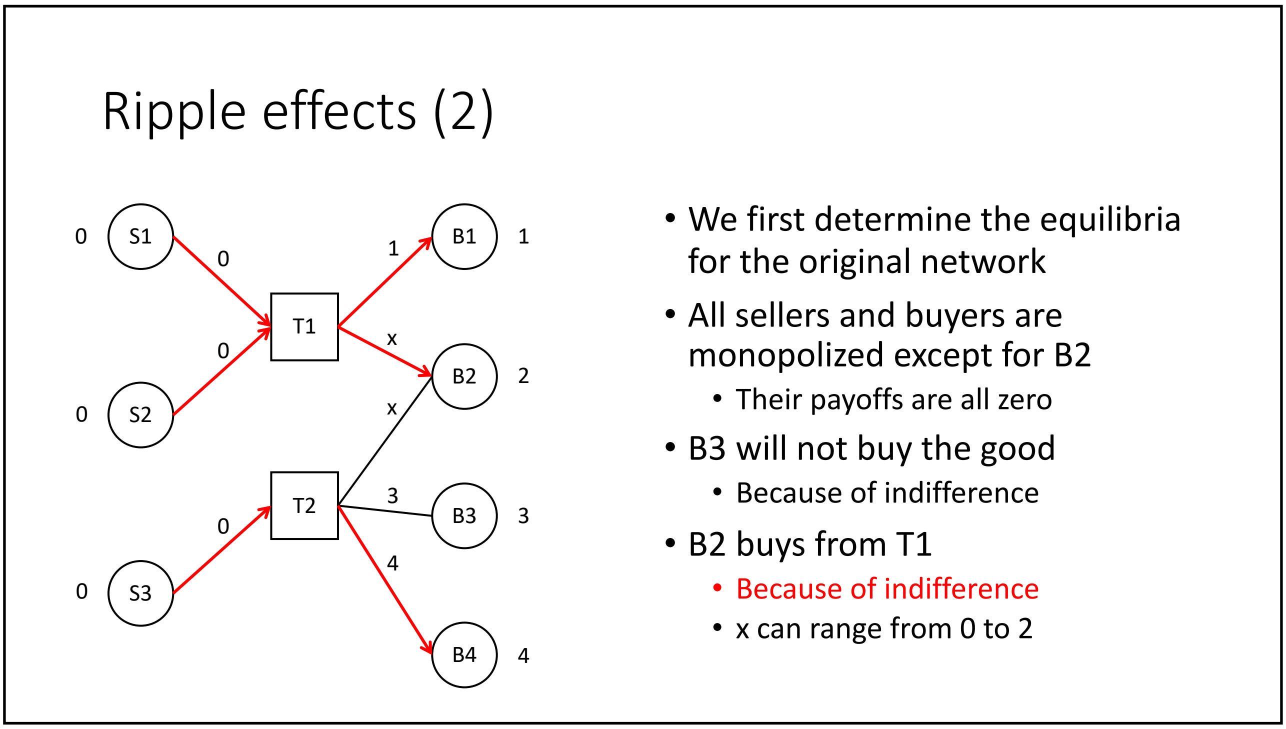


- T1’s strategy at equilibrium
  - T1 will sell to B1 at price w
  - T1 will buy from S1 at price x
- We use indifference
  - T1 and T2 have same bid x, but sale goes to T1
  - Remember how indifference works: when prices are pushed to the limit, we (the modelers) can break ties. You can assume that T1 has a price x+0.01, which is a tiny bit bigger than T2’s price.
- Second-price rule emerges naturally!
  - It wasn’t built into the system, it emerges from the equilibrium



Ripple effects (1)

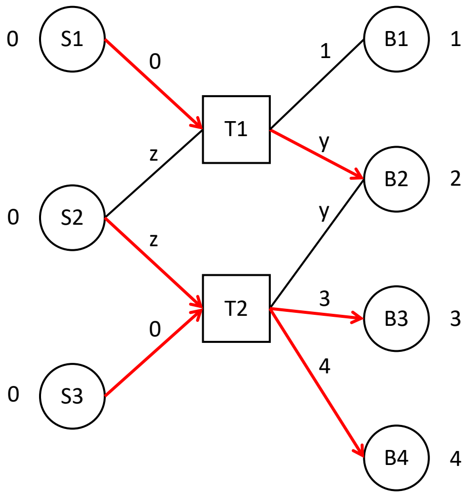
- Let’s use our model to explore how small changes to the network can affect payoffs
  - Including payoffs to nodes not involved in the change!
- We show how “shocks” to highly interconnected trading networks can propagate to more distant parts of the network
- Consider the network at the left
  - We will see what happens when we add the red edge
  - Can you work out the equilibria by yourself (before and after adding the red edge)? [exercise]



Ripple effects (2)

- We first determine the equilibria for the original network
- All sellers and buyers are monopolized except for B2
  - Their payoffs are all zero
- B3 will not buy the good
  - Because of indifference
- B2 buys from T1
  - Because of indifference
  - x can range from 0 to 2

Ripple effects (3)



- We determine the equilibria for the changed network
- The two bids to S2 must be the same, also the two asks to B2 must be the same
- S2 will sell to T2 rather than T1
  - To T1 price would be maximum 2 and T2 could outbid that
- The ask y must be at least 1
  - Also, y cannot be greater than 2
- T2 buys two copies, T1 one copy
- The bid z must be at least 1
  - Also, z cannot be greater than 3

High-level analysis of ripple effect

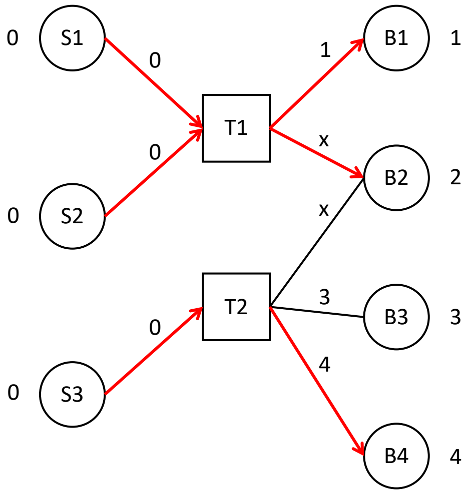
- In the original network, T2 has access to high-value buyers (they really want the good)
  - But T2's access to sellers is very limited, which reduces T2's profit
  - On the other hand, T1 uses all his trading opportunities
- In the changed network, S2 and T2 form a link
  - Buyer B3 now gets a copy of the good and B1 does not
  - The bottleneck has been removed: the high-value buyers can now buy the good
  - From B1's perspective, this is a nonlocal event: the link is formed between two nodes that are not neighbors of B1
- Other changes have occurred as well
  - Seller S2 is now in a much more powerful position and will get higher price (at least 1!)
  - Range of asks to B2 has been reduced from [0,2] to [1,2]: B2 benefited from weak sellers
- General conclusion: the network is partially under the control of the participants
  - For example, how much should S2 and T2 spend in order to create the link between them?

Social welfare
in trading networks

Social welfare in trading networks

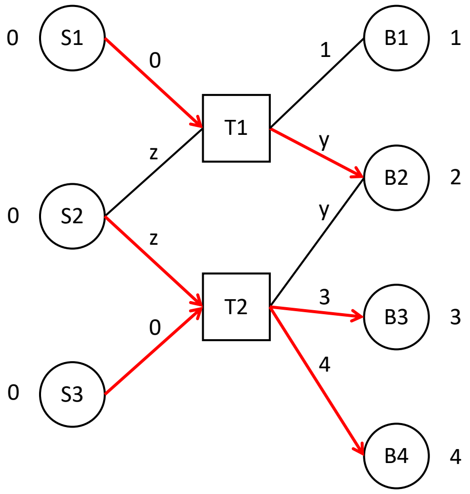
- What is the social welfare in our trading game?
  - Social welfare = sum of payoffs to all players
- Each good that moves from seller i to buyer j contributes $(v_j – v_i)$
  - How much more j values the good than i
  - Here is the computation: the good moves from i to t and then from t to j
    - $(b_{ti} – v_i) + (a_{tj} – b_{ti}) + (v_j – a_{tj}) = (v_j – v_i)$
- The social welfare is simply the sum of $(v_j – v_i)$ over all moved goods
  - How much happier the new owners are compared to the original owners
- Let’s see how our changed network affects the social welfare

Social welfare in original previous network



- The value of the social welfare is (1 + 2 + 4) = 7
  - This is independent of x

Social welfare in changed previous network



- The value of the social welfare is (2 + 3 + 4) = 9
  - This is independent of y or z
- Adding one edge has increased the social welfare
- A more richly connected network structure can enable greater social welfare from trade

Equilibria and social welfare

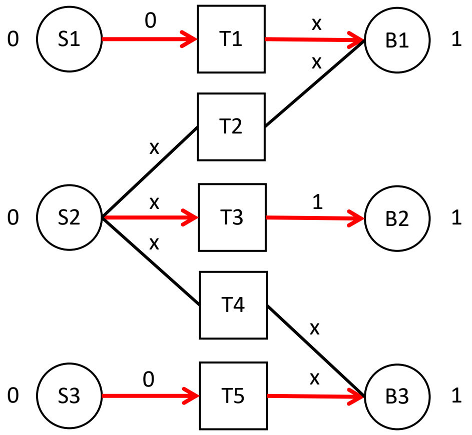
- The flow of goods that maximizes social welfare can be achieved by an equilibrium
  - Remember that each value of a variable (x, y, z, etc.) is an equilibrium
    - The values of the variables depend on factors outside of the network
  - There exist values for the variables that maximize social welfare
    - This picks an equilibrium
    - This statement can be proved, but we will not prove it
- In every trading network, there is always at least one equilibrium
  - This statement can be proved as well, but we will not prove it

Trader profits

Trader profits

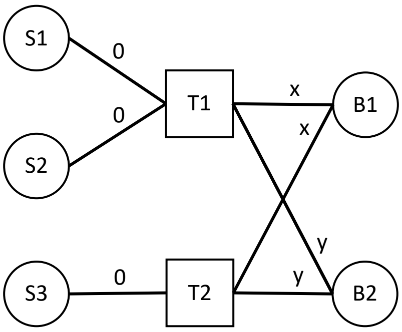
- How is the social welfare divided up (as payoffs) between sellers, buyers, and traders?
- We have the intuition that as the network becomes more richly connected, individual traders have less and less power!
- In order to make a profit (payoff>0), a trader must in some way be "essential" to the trading network
  - If there is one or more traders who can together replicate a trader's function, then that trader cannot make a profit
  - Let's take a look at some examples!

Trader profit example 1



- Any value of x from 0 to 1 results in an equilibrium
  - Traders T2 and T4 are left out, but their existence "locks" the value of x
- Traders don't always make a profit
  - When x=1, only T1 and T5 make a profit
  - When x=0, only T3 makes a profit
- All equilibria result in a social welfare of 3
  - The amount that goes to buyers and sellers (not traders) varies from 1 to 2 (for x from 1 to 0)

Trader profit example 2



- T1 and T2 both have monopoly power over their sellers
  - Yet, their profits are zero in every equilibrium
- If the common ask to either buyer were positive (>0) then the other trader could undercut it!
  - Therefore x=0 and y=0
  - “Race to the bottom”
  - The traders destroy each other’s profits!
- T1 fails to make a profit despite the fact that T2 can perform only one trade!
  - T2 threatens T1’s selling to both B1 and B2
  - Smaller trader T2 competes with larger trader T1, despite having insufficient sellers: “the threat is stronger than its execution”

Trader profit depends on essential edges

- For a given trader T in a network, does there exist an equilibrium in which T receives a positive payoff?
- There exists such an equilibrium precisely when T has an edge e to a seller or a buyer such that deleting e would change the social optimum
  - We say that e is an essential edge from T to the other node

Summary

Trading networks

- There are often intermediaries in markets
  - Intermediaries (traders) simplify management for buyers and sellers
  - We call the resulting graph a trading network
- We have defined an idealized trading network
  - Sellers on one side, traders in the middle, buyers on the other side
  - We can compute the equilibria of trading networks
  - Two important patterns are monopoly and perfect competition
- Our idealized trading network has many realistic phenomena
  - Implicit competition, connection to second-price auction, ripple effects
- Social welfare increases as connectivity increases
  - Also, trader profits decrease as connectivity increases!
  - Trader profit depends on essential edges (deleting them changes social optimum)

LINFO1115
Reasoning about a highly connected world

Lecture 8
Bargaining and power in networks

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Academic year 2022-23
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The course so far

Graph theory
Game theory
Lectures 1-4

- Triadic closure, homophily, structural balance, convergence
- Dominant strategies, Nash equilibria, social optimality

Applications of game theory
Lectures 5-8

- First applications
  - Traffic networks (drivers are players, routes are strategies)
    - Braess paradox
  - Auctions (one seller and many buyers)
    - First-price and second-price (most important)

generalize to

- Markets (many sellers and many buyers)
  - Matching markets (prices are the coordination mechanism)
  - Intermediaries (monopoly and competition: power depends on network!)
  - Bargaining with neighbors (node position: power depends on network!)

Today

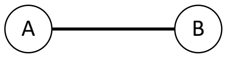
Bargaining and power in networks

Chapter 12

Negotiating in networks

- Negotiation in economic networks
    - When nodes are human beings or organizations, nodes negotiate with their neighbors
        - Node payoff depends on the node's position, the structure of the network, and human psychology
    - In today's lecture we will study negotiation in a network
        - We first introduce some experimental results
        - We then introduce three principles: Nash bargaining solution, Ultimatum Game, stable outcomes
        - We finally give a model to approximate real negotiations by humans: theory of balanced outcomes
- Node position was always important, in previous lectures too!
    - In our social networks, we saw the key role played by bridges
    - In markets with intermediaries, we saw that in a monopoly situation traders make all the profit and in a perfect competition the trader makes zero profit
- In today's course, we assume that nodes negotiate directly with their neighbors

Power in a social network

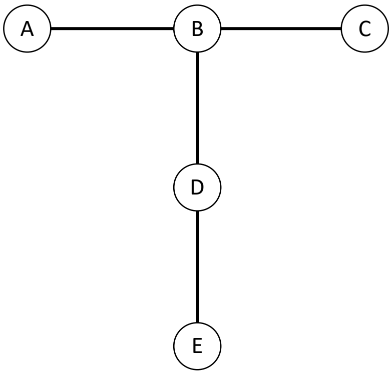


A and B work together, for example as buyer and seller. Which one gets higher payoff?

- Power is not a property of a person but a relationship with others
- Assume two people A and B work together in some way ("negotiate")
  - This work produces value for both of them (payoff, revenue, friendship, etc.)
  - The value can be divided equally or unequally
  - There might be an imbalance: one person gets more than the other
  - Power corresponds to imbalance: the more powerful person gets more
- Where does increased power of an individual come from?
  - It could be because of personality: the person is more talented
  - It could be because of position in the network: the person is in a dominant position

This is what we will study today

Example social network

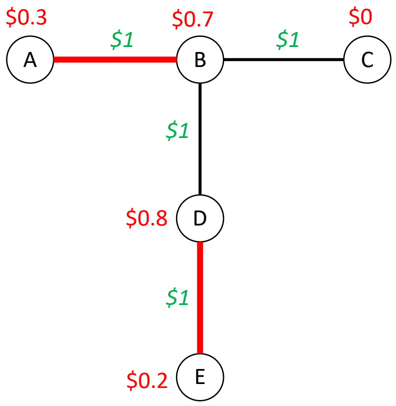


- Node B appears to hold a powerful position in this network, especially with respect to A and C. Why?
- Four reasons why B might get power:
  - Dependence: A and C are completely dependent on B for social relations
  - Exclusion: B has the ability to exclude A and C (but not D!)
  - Satiation: B will acquire value at a greater rate than the others, so B can place higher demands on value from its neighbors
  - Betweenness: Information can flow along paths, and B is on all paths starting at A or starting at C
- How valid are these reasons?
  - We will make experiments to study this

Network exchange model

- It is not obvious how a node gets power in a network
  - We will show how this has been studied and what has been learned
- Experimental setup for network exchange: (as always, we define and study an idealized model!)
  1. Choose a small graph and a test subject for each node. Each person can exchange instant messages with their neighbors.
  2. Put a fixed amount of money, say $1, on each edge, which can be divided between the two endpoints of the edge. This is called an exchange.
  3. Each node can exchange with only one neighbor (the one-exchange rule). The set of exchanges in one round is a matching in the graph (but not necessarily a perfect matching, since some nodes may not do an exchange).
  4. In one round, a node does simultaneous messaging with its neighbors, doing free-form negotiation. Negotiations must be concluded in a fixed time limit. As soon as the node reaches agreement with one neighbor, the negotiations with other neighbors are terminated.
  5. The experiment is run for multiple rounds. In each round, new money is placed on the edges (as in item 2), each node can do one exchange (as in item 3), and negotiate to divide money (as in item 4).

Example negotiation round



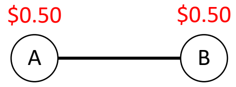
- Each edge starts with *$1* (in green)
- At the end of the round:
  - A-B have made an agreement
  - D-E have made an agreement
  - C has not made an agreement
- Results of the exchanges in red
  - Only two $1 values are exchanged
  - B gets $0.7, A gets $0.3
  - D gets $0.8, E gets $0.2
  - C gets $0 because no agreement
- What are the general rules for this kind of negotiation?
  - For a given network, who will get the most money? How can we figure this out?

Network exchange experiments

Negotiation experiments on networks

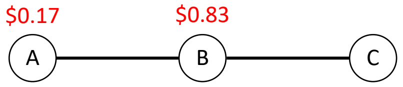
- To explain how to understand the negotiations on a network, we start by giving results of some experiments on human subjects
- These results are interesting, intuitive, and robust
  - We will use the insights from these results to lead to two important techniques to determine outcomes of negotations: the theory of stable outcomes and the theory of balanced outcomes
- We will start with simple networks and move to more complex ones
  - Paths with two, three, four, and five nodes
  - Five-node social network
  - Stem graph (triangle with a stub)
  - Pure triangle

The two-node path

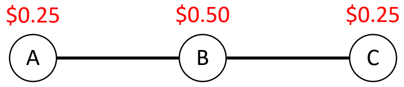


- The simplest network is a two-node path
- Two people are given a fixed amount of time in which to agree how to split $1
- What values will they agree on?
- Theoretical treatments usually predict an equal split
- Real experiments with human subjects align with this: approximately equal split

The three-node path



One-exchange rule

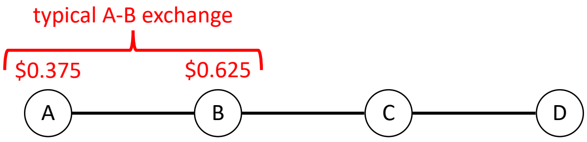


Two-exchange rule (variation)

- Intuitively, node B has power over both A and C
- As B negotiates with A, it has power to fall back on an alternative with C
  - Neither A nor C have other alternatives
- At least one of A or C is excluded from each round
  - Excluded subjects tend to ask for less in the next round
- The result is that B receives $0.83 (5/6)

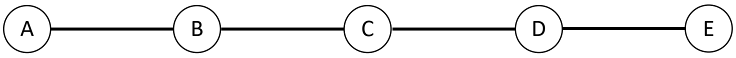
- A variation: if we allow two exchanges per round, then B negotiates on equal footing with A and C (B gets half on each agreement)

The four-node path



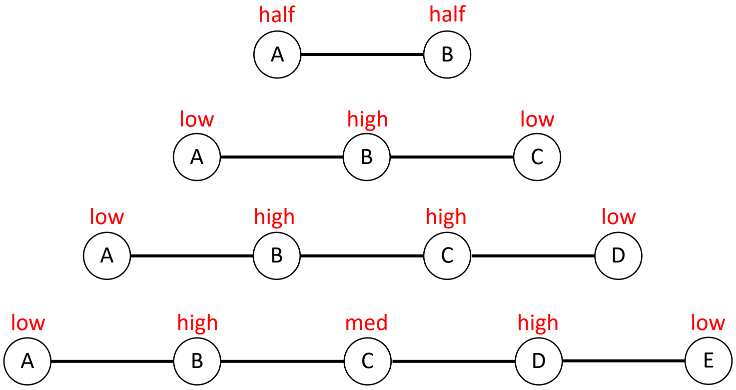
- The four-node path is much more subtle
  - In one outcome, we have A-B and C-D, but in another we have B-C
  - So B has some power over A, but it is weaker than with three nodes
  - If B excludes A, then B has to approach C who could make a deal with D
- B’s threat to A is a costly one to actually execute
  - In A-B exchanges, B gets from 7/12 ($0.58) to 2/3 ($0.67) of the money

The five-node path



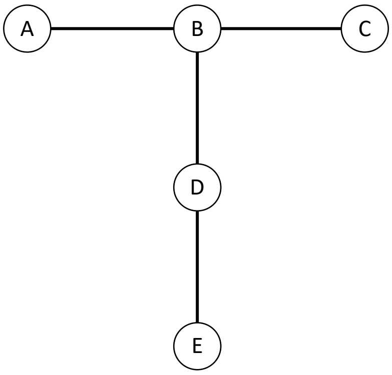
- The five-node path introduces a further subtlety
  - Node C occupies the central position, but it is quite weak (with one-exchange)
  - C’s only opportunities for exchange are with B and D, and each has alternative
    - C can be excluded almost as easily as A and E
- Experiments show that C does slightly better than A and E
  - C has much more power if it can exchange with both B and D in one round

Summary of power in paths



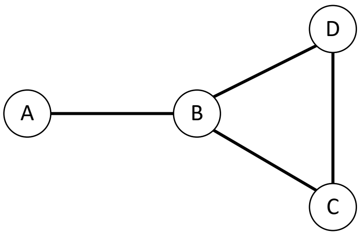
- This diagram gives the approximate power for each node in straight networks from two to five nodes

Our original social network



- B has the ability to exclude both A and C
  - B has most power, A and C are weak
- B and D almost never exchange
  - D and E have roughly equal power

“Stem graph”



- C and D will usually agree
  - C and D have similar power
- A and B will usually agree
  - B is slightly more powerful than A (because there is a small chance B will exclude A)
- Node B in stem graph makes slightly more money than node B in four-node path
  - Because B’s threat in four-node is to negotiate with (powerful) C; here B’s threat is to negotiate with weaker nodes

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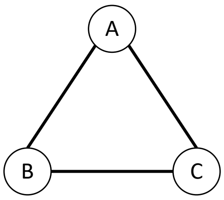
A pathological network



- All our networks so far have had reliable, convergent negotiations
- There exist pathological networks in which negotiations drag out until the very end
  - There is no convergence to an agreement
- In this triangle, only one exchange can be done
  - If A and B are agreeing, then C will butt in at the end
  - This will happen for all potential agreements
  - The infinite cycle is only stopped by the time limit
- Stem graph contains a triangle but converges!
  - Stem graph converges because of the fourth node

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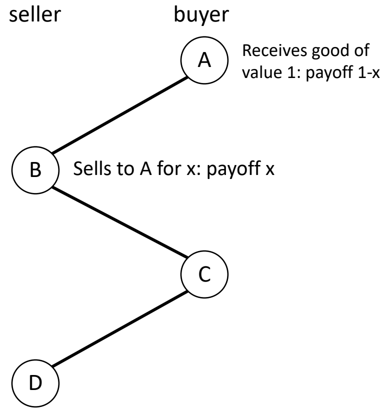
Connection to buyer-seller networks

Correspondence between matching markets and exchange networks



- In bipartite matching markets (chapter 10) we have buyers and sellers
  - In negotiation networks all nodes play the same role (no division into buyers and sellers)
  - Despite this apparent difference, there is a close connection!
- Consider the four-node path
  - It corresponds to the bipartite market at the left
  - Negotiation between A and B over price x is exactly the same as negotiation between A and B over division of $1
  - One-exchange rule corresponds to selling a single unit
- Caveat is that this only works for bipartite graphs
  - And it is only true at a mathematical level: humans would not necessarily behave the same way in both cases!

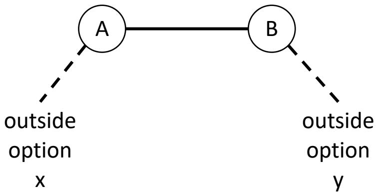
Toward a general framework for evaluating negotiations

A mathematical framework for negotiation

- We have developed some informal motivations for negotiations
  - Let us now move toward a more mathematical framework
- We first introduce three important principles
  - The Nash bargaining solution
  - The Ultimatum Game
  - Stable outcomes
- Using these principles, we can define a negotiation framework
  - Theory of balanced outcomes: outcome is balanced when all agreements are Nash bargaining solutions; balanced outcomes exist whenever stable outcomes exist
- With the theory of balanced outcomes, we can approximate the results of human negotiations (except possibly for very complex networks)

Two-person interaction:
the Nash bargaining solution

The Nash bargaining solution



- Assume that A and B are negotiating over how to split \$1
- We extend the scenario by assuming A has an outside option of x and B has an outside option of y
  - If A does not like his share of the \$1, he can always take x
- We will assume $x+y \leq 1$ (else no agreement is possible!)
  - Given this, A will ask for at least x and B will ask for at least y
  - So the negotiation is really about splitting the surplus $s=1–x–y$ (and we know $s \geq 0$)
- So we predict that they will split s in half
  - A gets $x+(s/2)$ and B gets $y+(s/2)$
  - A gets $(x+1–y)/2$
B gets $(y+1–x)/2$ Nash bargaining solution
- This is reasonable: negotiate with a strong outside option!

Two-person interaction: the Ultimatum Game

The Ultimatum Game

- The Ultimatum Game models what happens when A and B negotiate with an extreme power imbalance
- We assume two people divide $1 as follows:
  - A is given a dollar and is told to propose a division with B, that is, A proposes how much to keep himself and how much to give to B.
  - B is given the option to approve or reject the division.
  - If B approves, both get the proposed amount. If B rejects, both get zero.
- We assume this is a one-shot interaction (no further rounds)
- How should they behave?
  - Game theory says B should accept if the proposed amount > 0 (even very small!)
  - But humans do not behave this way!

How humans play the Ultimatum Game

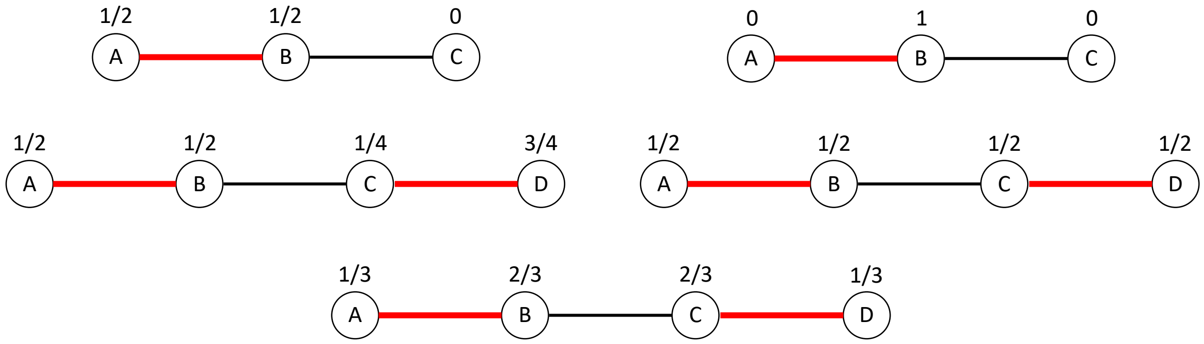
- Experiments were done with human subjects playing this game
  - Typically, A offered fairly balanced divisions (around 2/3 versus 1/3)
  - Very unbalanced offers were usually rejected by B (if A offers $0.01)
- These results are robust even with very large amounts of money
- How can we explain this in the context of game theory?
  - Payoff includes an emotional payoff: the feeling of being cheated or not
  - B finds greater benefit in rejecting a low offer and feeling good about it
- Conclusion
  - Humans will not split extreme imbalances as widely as basic models predict
  - When the model gives extreme values, they should be reduced for humans

Network interactions: stable outcomes

Stable outcomes in networks

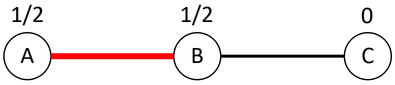
- The concept of stable outcomes is one way to approximate the outcomes of negotiations on arbitrary graphs
- Definition of an outcome of a network exchange on a graph:
  - First, a matching on the set of nodes. This corresponds to the one-exchange rule, and some nodes may be left out.
  - Second, a number associated with each node. This value indicates how much the node gets from the exchange. If two nodes are matched, the sum of values should be 1. If a node is not in a matching, its value should be 0.
- Outcomes should be stable in the following sense:
  - No node X can propose an offer to Y such that both X and Y are better off (X has "stolen" Y away from an existing agreement)

Some examples of outcomes

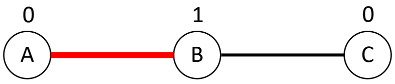


- Here are some examples of outcomes on three- and four-node paths
- As we will see, some of these are stable and some are unstable!
  - Figure out which are stable and which are unstable, without looking at other slides [exercise]

Some outcomes have instability



Unstable, because C can offer 2/3 to B
and keep 1/3, so both are better off!



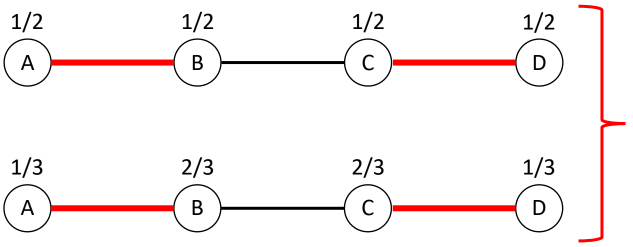
Stable, because there is nothing C
can offer to B to improve B's value!

- We should expect outcomes to be stable: no node X can propose an offer to a neighbor node Y such that both X and Y are better off
  - Basically, "stealing" node Y away from an existing agreement (sabotaging the agreement)
  - This situation is called an instability
- Definition of an instability: Given an outcome consisting of a matching and set of values, an instability in this outcome is an edge not in the matching, joining two nodes X and Y, such that (X's value) + (Y's value) < 1

Usefulness of stable outcomes

- For an outcome not to have an opportunity for two nodes X and Y to disrupt the situation, the outcome must be stable, defined as follows
- Definition of a stable outcome: An outcome of a network exchange is stable if and only if it contains no instabilities
- We expect to see stable outcomes in human experiments!
- Stable outcomes capture some of the general principles of network exchange experiments
  - They capture well the effects of extreme power imbalances (why B gets 1)
    - Outcomes like 1/6 – 5/6 are as close to extremes 0 – 1 as humans will get
  - They capture well the pathological cases like the free-standing triangle

Limitation of stable outcomes



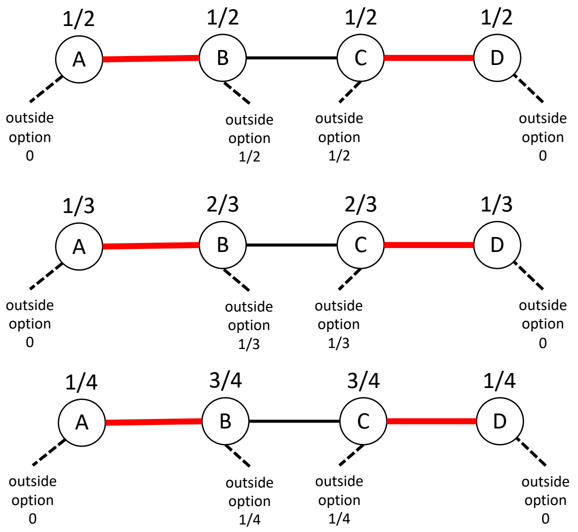
Both outcomes are stable

Any outcome is stable when (B's value) + (C's value) ≥ 1

- When there is a weak power imbalance, stable outcomes are too ambiguous
  - For example, in these two four-node paths, there is a large range of stable outcomes
- Stability is too weak in networks with small power differences
  - Can we strengthen stability so it picks the outcomes most likely in real life?

The theory of balanced outcomes

Balanced outcomes for the four-node path

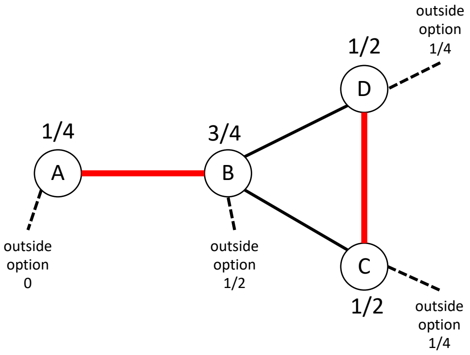


- In the case when there are many possible stable outcomes for a network, we will show how to select the best of the stable outcomes
- Compare the three examples on this slide
  - All three are stable, but they're not equally good!
  - We use the formulas for Nash bargaining solution
- In the first example, A-B is not a Nash bargain
  - (1/2 + 1)/2 to B, (–1/2 + 1)/2 to A ⇒ 3/4, 1/4
  - This example is unbalanced
- In the second example, A-B (and C-D) are both Nash bargaining solutions!
  - (1/3 + 1)/2 to B, (–1/3 + 1)/2 to A ⇒ 2/3, 1/3
  - This example is balanced: it is the best of the three
- In the third example, A-B is not a Nash bargain
  - (1/4 + 1)/2 to B, (–1/4 + 1)/2 to A ⇒ 5/8, 3/8
  - This example is unbalanced

The theory of balanced outcomes

- Definition of a balanced outcome:
  - An outcome (consisting of a matching plus node values) is balanced, if for each edge in the matching, **the money split represents the Nash bargaining solution** for the two nodes involved, given the best outside options for each node provided by the values in the rest of the network
- In our example, this prevents B and C from getting too little or too much
  - They will get exactly the values of their Nash bargaining solution
- Note that this definition is recursive: the values of nodes B and C depend on their outside options, which depend on the values of the other nodes, which depend on the others' outside options, which depend on the values of nodes B and C
  - Calculating a balanced outcome means solving these recursive equations, usually not hard
  - We can show that any network with a stable outcome also has a balanced outcome
- Do balanced outcomes always mirror how humans negotiate? Great question!
  - This is actually not known for large networks; it is still a research topic

Balanced outcomes for the stem graph



- The stem graph is another example of balanced outcomes
- C and D exchange on equal terms
  - This gives outside option of 1/2 for B
- This leads to a Nash bargaining solution of 1/4 – 3/4 for A and B
  - A gets (x+1–y)/2 = (0+1–1/2)/2 = 1/4
    B gets (y+1–x)/2 = (1/2+1-0)/2 = 3/4
- This corresponds approximately to what humans do in experiments

Conclusion for balanced outcomes

- Balanced outcomes can be computed in practice
  - A balanced outcome exists whenever a stable outcome exists
  - There are methods to find all balanced outcomes for any network
- As far as we know, balanced outcomes correspond to what humans do
  - There is no proof of this, and it is still a research question for large graphs
  - There are competing theories that achieve similar results
- For further study, you can look at cooperative game theory
  - How a collection of players will divide up the value that comes from a collective activity (such as network exchange in our case)

Summary

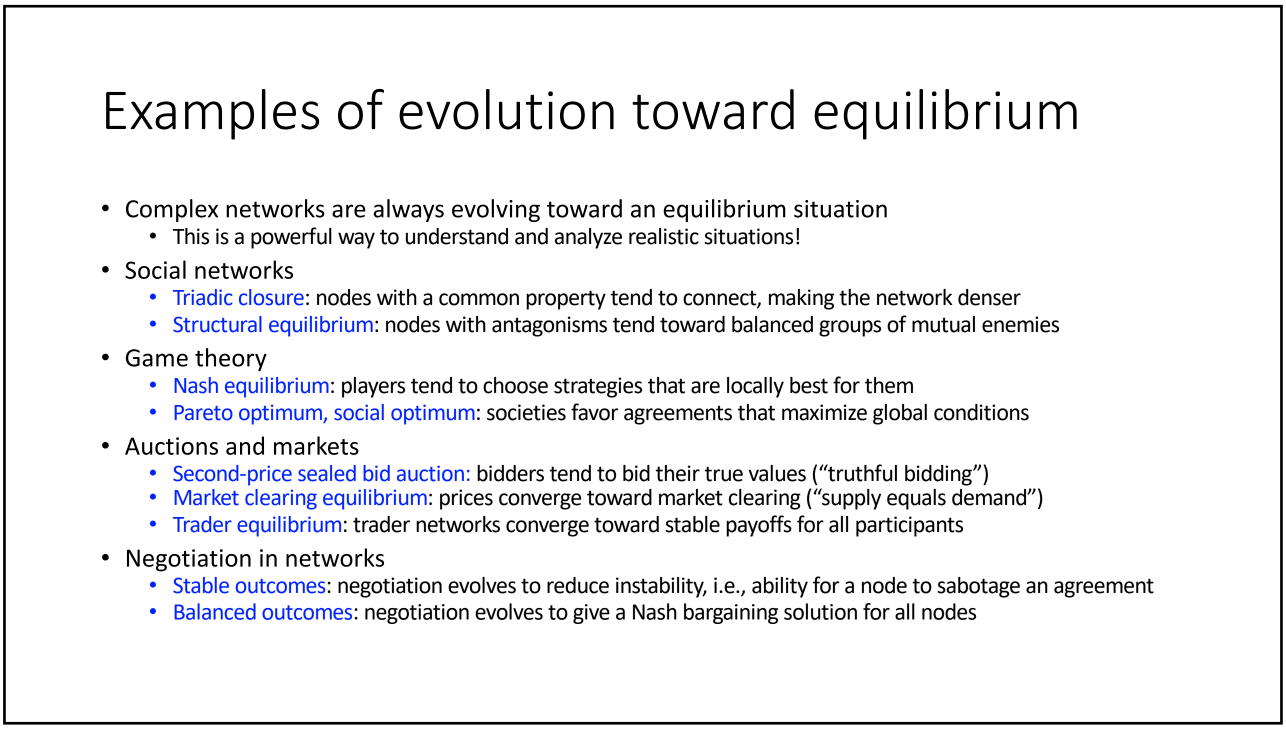
Negotiation on a network

- How do nodes negotiate when they are a network?
- Their negotiating power depends on their position in the network
  - Each node negotiates only with its neighbors
  - It is not obvious how this works, so experiments were done with humans and theoretical models were devised
- Three important principles
  - Nash bargaining solution: all agreements are considered fair by their participants
  - Ultimatum game: humans stay away from extreme imbalances when negotiating a deal
  - Stable outcomes: agreements cannot be “stolen” by another node offering a better deal
- Proposed model
  - Theory of balanced outcomes: each edge in the matching gives the Nash bargaining solution
  - Every balanced outcome is stable and every network with a stable outcome has a balanced outcome
  - This seems to correspond with how humans negotiate in a network!

Main theme of the course: evolution toward equilibrium

Course organization and main theme

- Organization of the course
  - First part (four lectures): Introduction to graph theory and game theory
  - Second part (four lectures): Focus on game theory for markets and negotiation
  - Second part (five lectures): Focos on graph theory for real networks, namely Web, search, information networks, economic networks
- Overarching theme of the course
  - Complex situations are always evolving toward some equilibrium
  - We have given idealized models of real situations and proved this
  - The idealized models are precise and permit both to understand and compute
  - The intuitions of the idealized models still hold in realistic adaptations
  - This gives you powerful tools to analyze realistic situations on networks!



Examples of evolution toward equilibrium

- Complex networks are always evolving toward an equilibrium situation
  - This is a powerful way to understand and analyze realistic situations!
- Social networks
  - Triadic closure: nodes with a common property tend to connect, making the network denser
  - Structural equilibrium: nodes with antagonisms tend toward balanced groups of mutual enemies
- Game theory
  - Nash equilibrium: players tend to choose strategies that are locally best for them
  - Pareto optimum, social optimum: societies favor agreements that maximize global conditions
- Auctions and markets
  - Second-price sealed bid auction: bidders tend to bid their true values ("truthful bidding")
  - Market clearing equilibrium: prices converge toward market clearing ("supply equals demand")
  - Trader equilibrium: trader networks converge toward stable payoffs for all participants
- Negotiation in networks
  - Stable outcomes: negotiation evolves to reduce instability, i.e., ability for a node to sabotage an agreement
  - Balanced outcomes: negotiation evolves to give a Nash bargaining solution for all nodes

LINFO1115
Reasoning about
a highly connected world

Lecture 9
World Wide Web: structure, search, and evolution

Peter Van Roy

Academic year 2022-23
École Polytechnique de Louvain
Université catholique de Louvain

World Wide Web

- The World Wide Web is the most important information and economic network overlaid on the Internet
- Structure of the Web
  - The Web is a fully decentralized information graph distributed over the Internet
  - The Web has an emergent bow-tie structure with a core connected component
- Searching information on the Web
  - The Web introduces an information retrieval problem based on abundance
  - Link analysis has solved this problem using hubs and authorities (PageRank)
- Evolution of the Web
  - The modern Web is based on content, services, and people (Web 2.0)
  - Modern search engines introduce targeted advertising and adversarial problems

The structure of the Web

Chapter 13

Overlay networks on the Internet

- The Internet provides world-wide network connectivity
  - A scalable, decentralized transport network with symbolic names
  - Basic transport protocols are IP (unreliable packets) and TCP (reliable streams)
  - Basic symbolic addressing protocol is DNS
- All other networks are overlaid on top of the Internet
  - Used as information networks or economic networks
  - Facebook, Twitter, etc.: social networks for human beings
  - World Wide Web: hypertext graph for human information and interaction
- Today we will study the World Wide Web
  - The most important information network overlaid on the Internet

World Wide Web overview

- Hypertext graph distributed over Internet nodes
  - Combines two separate ideas, hypertext and network distribution
  - Hypertext: each node is a page that contains links to other pages
  - Network distribution: pages are geographically distributed at Internet nodes
- Originally developed to let people share information over the Internet
  - Original authors Tim Berners-Lee and Robert Cailliau at CERN (1989-1991)
- Key features
  - Publication of documents as Web pages on your computer
  - Access to Web pages using a software tool (browser) on your computer
  - Completely decentralized: individual users both create and access Web pages
  - Implementation defines a new protocol, HTTP, overlaid on TCP

Precursors

Citations as a precursor to hypertext

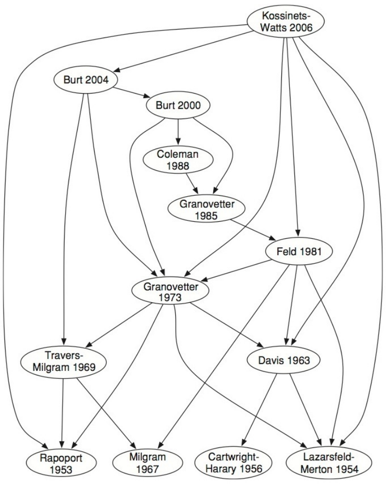


Figure 13.3. The network of citations among a set of research papers forms a directed graph that, like the Web, is a kind of information network. In contrast to the Web, however, the passage of time is much more evident in citation networks, since their links tend to point strictly backward in time.

- A scholarly work includes citations to credit earlier work
  - Scientific papers, patents, legal decisions
- Citations are governed by an "arrow of time": citations are frozen when an article is written (into the past)
  - There exist rare exceptions to this principle, for example when two papers are published at the same time
- The Web is not frozen, but continuously evolving, with links to the most recent version of an evolving document

Encyclopedias as a precursor

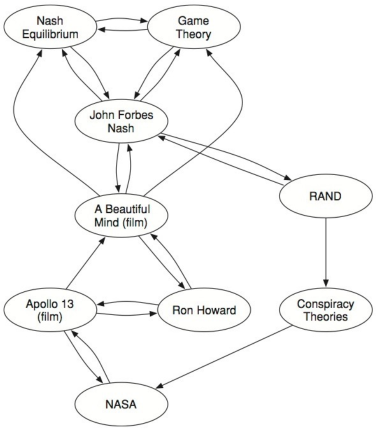
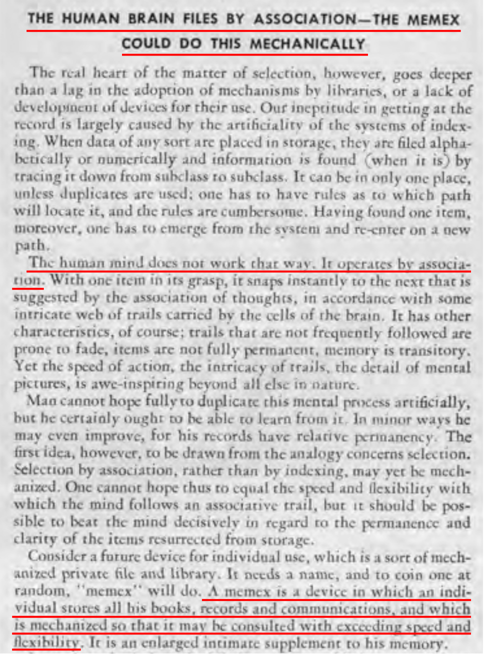


Figure 13.4. The cross-references among a set of articles in an encyclopedia form another kind of information network that can be represented as a directed graph. The figure shows the cross-references among a set of Wikipedia articles on topics in game theory and their connections to related topics, including popular culture and government agencies.

- This diagram shows cross-references among Wikipedia articles on game theory and connections to related topics
  - Note how there are multiple paths between "Nash Equilibrium" and "NASA"
- This is an associative memory
  - Random-access memory looks up content from its address; an associative memory looks up content according to the concepts in the document
  - Browsing through chains of cross-references is similar to mental free association, where one concept leads to another
- A related concept is a semantic network
  - Nodes are concepts and edges are relationships between concepts

Vannevar Bush and the Memex

- Vannevar Bush, “As We May Think”, Atlantic Monthly, 1945
  - Vannevar Bush was a scientist, technologist, and American national science advisor up to World War 2
- Memex device: digitized versions of human knowledge connected by associative links
  - Motivated by vision of growth of computer and information technology
- Foreshadowed many aspects of the Web
  - Encyclopedia, socioeconomic system, aid to logical reasoning (“global brain”)



THE HUMAN BRAIN FILES BY ASSOCIATION—THE MEMEX COULD DO THIS MECHANICALLY

The real heart of the matter of selection, however, goes deeper than a lag in the adoption of mechanisms by libraries, or a lack of development of devices for their use. Our ineptitude in getting at the record is largely caused by the artificiality of the systems of indexing. When data of any sort are placed in storage, they are filed alphabetically or numerically and information is found (when it is) by tracing it down from subclass to subclass. It can be in only one place, unless duplicates are used; one has to have rules as to which path will locate it, and the rules are cumbersome. Having found one item, moreover, one has to emerge from the system and re-enter on a new path.

The human mind does not work that way. It operates by association. With one item in its grasp, it snaps instantly to the next that is suggested by the association of thoughts, in accordance with some intricate web of trails carried by the cells of the brain. It has other characteristics, of course; trails that are not frequently followed are prone to fade, items are not fully permanent, memory is transitory. Yet the speed of action, the intricacy of trails, the detail of mental pictures, is awe-inspiring beyond all else in nature.

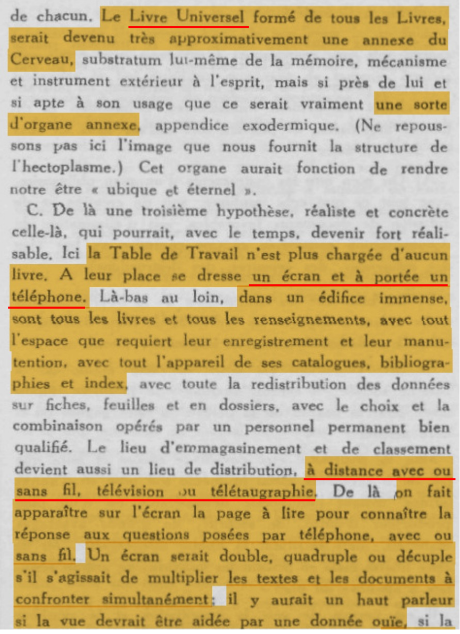
Man cannot hope fully to duplicate this mental process artificially, but he certainly ought to be able to learn from it. In minor ways he may even improve, for his records have relative permanency. The first idea, however, to be drawn from the analogy concerns selection. Selection by association, rather than by indexing, may yet be mechanized. One cannot hope thus to equal the speed and flexibility with which the mind follows an associative trail, but it should be possible to beat the mind decisively in regard to the permanence and clarity of the items resurrected from storage.

Consider a future device for individual use, which is a sort of mechanized private file and library. It needs a name, and to coin one at random, ''memex'' will do. A memex is a device in which an individual stores all his books, records and communications, and which is mechanized so that it may be consulted with exceeding speed and flexibility. It is an enlarged intimate supplement to his memory.

Paul Otlet and the Universal Book

- Paul Otlet, “Traité de Documentation: Le Livre sur le Livre”, 1934
  - Paul Otlet was a documentalist, entrepreneur and peace activist in the period before World War 2
- Universal book: all human knowledge collected together, in a continuous collaborative effort, accessible by all from their homes through a network
  - Motivated by a desire to achieve peace through universal dissemination of knowledge (19th century utopian vision)
- Foreshadowed networked information systems and search engines

Universal Book



de chacun. Le Livre Universel formé de tous les Livres, serait devenu très approximativement une annexe du Cerveau, substratum lui-même de la mémoire, mécanisme et instrument extérieur à l'esprit, mais si près de lui et si apte à son usage que ce serait vraiment une sorte d'organe annexe, appendice exodermique. (Ne repoussons pas ici l'image que nous fournit la structure de l'hectoplasme.) Cet organe aurait fonction de rendre notre être « ubique et éternel ».

C. De là une troisième hypothèse, réaliste et concrète celle-là, qui pourrait, avec le temps, devenir fort réalisable. Ici la Table de Travail n'est plus chargée d'aucun livre. A leur place se dresse un écran et à portée un téléphone. Là-bas au loin, dans un édifice immense, sont tous les livres et tous les renseignements, avec tout l'espace que requiert leur enregistrement et leur manutention, avec tout l'appareil de ses catalogues, bibliographies et index, avec toute la redistribution des données sur fiches, feuilles et en dossiers, avec le choix et la combinaison opérés par un personnel permanent bien qualifié. Le lieu d'emmagasinement et de classement devient aussi un lieu de distribution, à distance avec ou sans fil, télévision ou télétaugraphie. De là on fait apparaître sur l'écran la page à lire pour connaître la réponse aux questions posées par téléphone, avec ou sans fil. Un écran serait double, quadruple ou décuple s'il s'agissait de multiplier les textes et les documents à confronter simultanément; il y aurait un haut parleur si la vue devrait être aidée par une donnée ouïe, si la

Screen and telephone

Networked, multimedia

Bow-tie structure of the Web

Web as directed graph

- Information networks are usually directed, whereas social networks are undirected
  - Global friendship network is undirected
    - Edge between A and B if A and B know each other
  - Global name-recognition network is directed
    - Edge from A to B if A has heard of B

- Definitions for directed graph
  - A path from A to B is a sequence of nodes that starts with A and ends with B, such that each pair is a directed edge in the graph
  - A directed graph is strongly connected if there is a path from every node to every other node

Strongly connected components

- Many directed graphs are not strongly connected, yet it is important to understand how nodes can be reached by other nodes
  - In an undirected graph, we can partition the nodes into connected components: two nodes in the same connected component can reach each other, so the situation is symmetric
  - It is not so simple in a directed graph, because connectivity is not symmetric
- We define a strongly connected component (SCC) in a directed graph as a subset of nodes such that:
  1. Each node in the subset has a path to each other node in the subset
  2. The subset is maximal, i.e., it is not part of a larger subset with the property that each node can reach the other
- A directed graph can be translated into an acyclic directed graph between its strongly connected components
  - Each SCC is a “supernode”, that is, a node in the translated graph

Graph of the strongly connected components

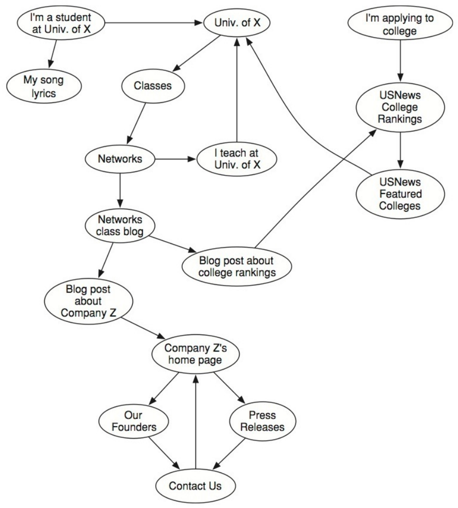


Figure 13.5. A directed graph formed by the links among a small set of Web pages.

We convert a directed graph into an acyclic directed graph of its strongly connected components

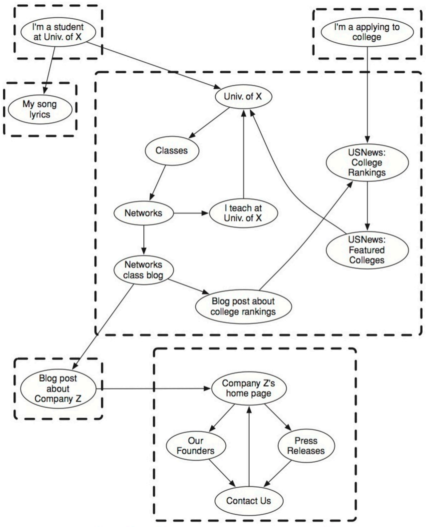


Figure 13.6. A directed graph with its strongly connected components identified.

Bow-tie structure of the Web

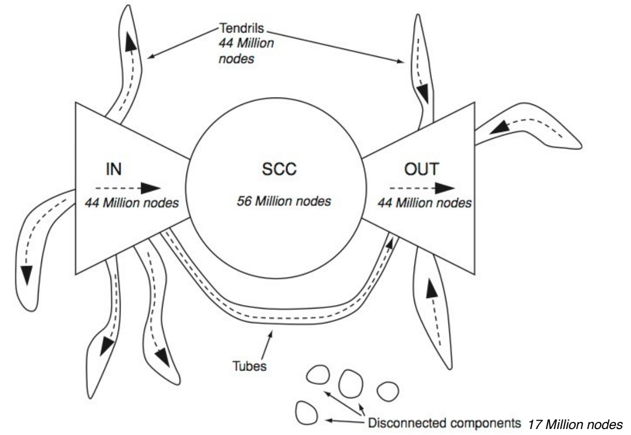


Figure 13.7. A schematic picture of the bow-tie structure of the Web (image from Broder et al., [80]). Although the numbers are now outdated, the structure has persisted.

- Andrei Broder built a global map of the Web, using strongly connected components as building blocks
  - An abstract map that shows the structure of the Web
- They found that the Web contains a giant strongly connected component
  - There is almost surely only one giant SCC. If there were two, X and Y, it would take only two links, one from X to Y and one from Y to X, to merge them.
- The structure continues to exist despite the enormous growth of the Web since 1999
  - The level of abstraction has now increased, however, giving a new structure called the Web 2.0

Web 2.0

Emergence of Web 2.0

- The original Web was a set of linked text pages, such that the pages were mostly static and new pages were added to the set
- During the period 2000 to 2009, the abstraction level increased
  i. Collaborative creation and maintenance of shared content, instead of individuals creating pages (content = a dynamic subgraph of pages that are conceptually related)
  ii. Creation of services that manage data, instead of sets of pages (service = software that conceptually manages a part of the Web)
  iii. Connections between people instead of between pages (pages are identified by their owners, not just by their link URLs)
- Examples
  - Wikipedia: a collective encyclopedia (principle (i))
  - Gmail and other online e-mail: common management of e-mail (principle (ii))
  - MySpace and Facebook: each person has a personal Web graph (principle (iii))

Social phenomena in Web 2.0

- *Software that gets better the more people use it*: Online services are more useful and more appealing as more people use them. eBay is more useful as more people put objects for sale and more buy objects. Recommendation systems and trust systems are more useful as more people use them.
  ➜ Direct-benefit cascades (later in the course)

- *The wisdom of crowds*: Collaborative authorship on Wikipedia, group evaluation of news content on Digg, breaking news on Flickr and Twitter. This process can also fail, because of herding and instability.
  ➜ Network cascades (later in the course)

- *The "Long Tail"*: Combining a small amount of very popular content with a huge amount of less popular content with "niche appeal". The less popular content is often bigger than the popular content.
  ➜ Power laws and "rich-get-richer" phenomena (later in the course)

Link analysis and Web search

Chapter 14

The problem of search

- Information retrieval finds documents with keyword queries
- This has many problems
  - Inexpressiveness: subtle ideas cannot be expressed using keyword combinations
  - Synonymy: multiple words for the same concept, “scallions” vs. “green onions”
  - Polysemy: the same word has multiple meanings, “jaguar” gives cars instead of animals
- Information retrieval was formerly a job of specialists
  - But now, with decentralization everyone is both author and searcher
  - But now, the quantity of information is enormous and growing exponentially
  - But now, information is constantly changing in content and nature
    - On Sep. 11, 2001, typing “World Trade Center” in Google did not show the terrorist attack, but the building
  - How can we solve these problems?
- There is another, bigger problem: there is too much relevant information
  - Search has transformed from a scarcity problem to an abundance problem

How can we find the right page?

- Say we give the one-word search “Cornell”
    - What are the clues that will direct us to Cornell University’s home page?
- First idea: voting by in-links
    - A key idea is that there is nothing on the page itself that can direct us: it does not use the word “Cornell” any differently than thousands of other pages
    - The right page stands out because other pages link to it
    - So we gather pages relevant to Cornell, and find the page with most in-links
- In-links are not good enough, though
    - Some pages will always have many in-links, simply because they are popular!
    - So what can we do?

Example of using in-links

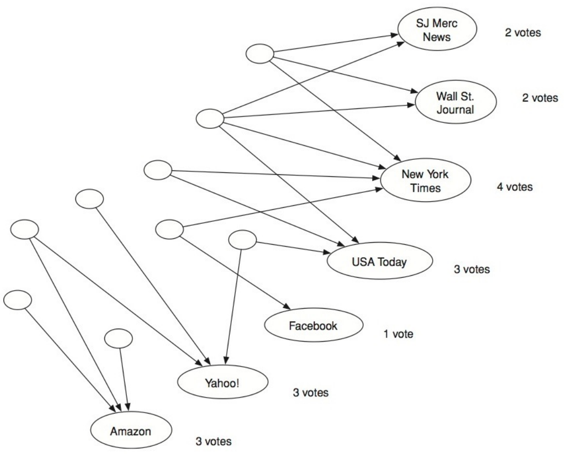


Figure 14.1. Counting in-links to pages for the query “newspapers.”

- We count in-links to pages for the query “newspapers”
    - We start by collecting pages that are relevant to the query “newspapers”
    - We then see which pages get the most in-links from the collected pages
- Newspapers New York Times and USA Today get a lot of votes
- But Amazon and Yahoo! get a lot of votes too even though they are not newspapers (they are popular)
    - How can we remove them?
- Luckily, the network has more information that we can use
    - Pages that have lists of newspapers

Finding lists of resources

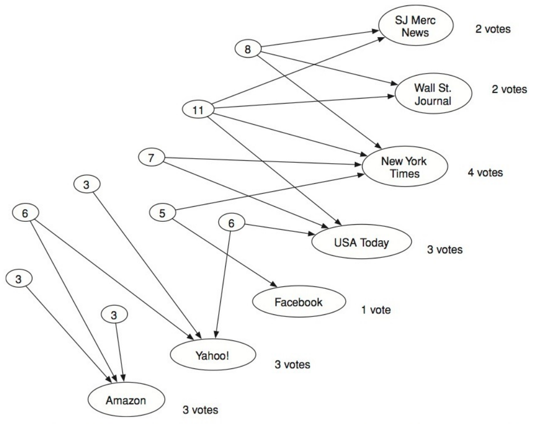


Figure 14.2. Finding good lists for the query "newspapers": each page's value as a list is written as a number inside it.

- We improve the approach by using lists
  - A list is a page that has a list of resources
  - Many such pages exist, they can say "here are some good links to follow"
- How can we find good lists?
  - Look at the pages that link to the pages with the highest number of votes
  - These pages link to the good ones!
- A page's score as a list = sum of votes for all the pages it links to

Principle of repeated improvement

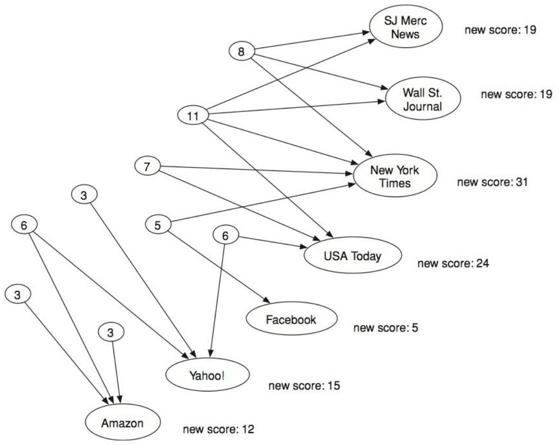


Figure 14.3. Reweighting votes for the query "newspapers": each labeled page's new score is equal to the sum of the values of all lists that point to it.

- We have now found the best lists
- Let's use them to reweight the votes
  - A link from a best list counts more than a link from another page
- We can repeat to keep improving
  - Find the pages with most votes
  - Find the best lists, which link to them
  - Reweight the pages with most votes
  - Reweight the best lists to them
  - And keep going! When does this end?

Normalizing the number of votes

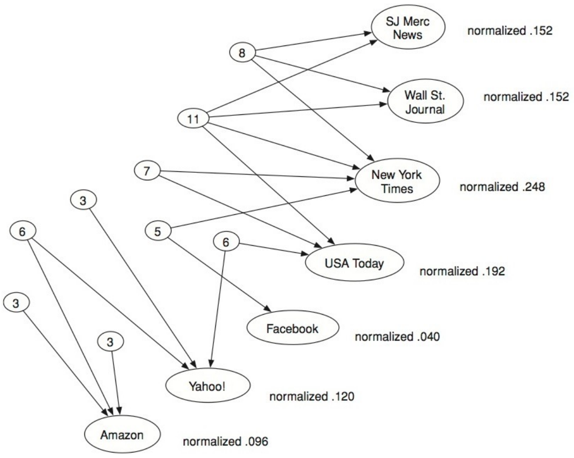


Figure 14.4. Reweighting votes after normalizing for the query "newspapers."

- To keep the numbers from getting large, we normalize each page's score by dividing by the sum of all page scores
- This changes the scores of the best lists too
- We use repeated improvement and normalize after each step
  - When does this end?

Hubs and authorities

- We introduce two concepts, hubs and authorities
  - Authority: a page we are looking for, a high-quality answer to the query
  - Hub: a page that has a high-quality list to the pages we are looking for
- We define two numerical scores, auth(p) and hub(p), giving values for page p
- Algorithm to compute hub and authority scores:
  - Start with all hub and authority scores equal to 1
  - Repeat k times:
    - For each page p, update auth(p) to be sum of all hub pages pointing to it
    - For each page p, update hub(p) to be sum of all authority pages it points to
    - Normalize the authority scores by dividing by sum of all authorities (and also for hubs)
- We can show that the normalized values converge to limits
  - Limiting values are independent of the initial values of hub and authority scores

Hubs and authorities algorithm

- We give a formal definition of the algorithm
- Initialization
  - $\forall p: \text{auth}(p):=1$
    $\forall p: \text{hub}(p):=1$
- Iteration: *(repeated k times)*
  - Update step: *(all values updated simultaneously)*
    $\forall p: \text{auth}(p) := (\sum_{p' \text{ such that } (p' \rightarrow p)} \text{hub}(p'))$
    $\forall p: \text{hub}(p) := (\sum_{p' \text{ such that } (p \rightarrow p')} \text{auth}(p'))$
  - Normalization step: *(all values updated simultaneously)*
    $\forall p: \text{auth}(p) := \text{auth}(p) / \sum_{\text{all pages } p} \text{auth}(p)$
    $\forall p: \text{hub}(p) := \text{hub}(p) / \sum_{\text{all pages } p} \text{hub}(p)$

Limiting values

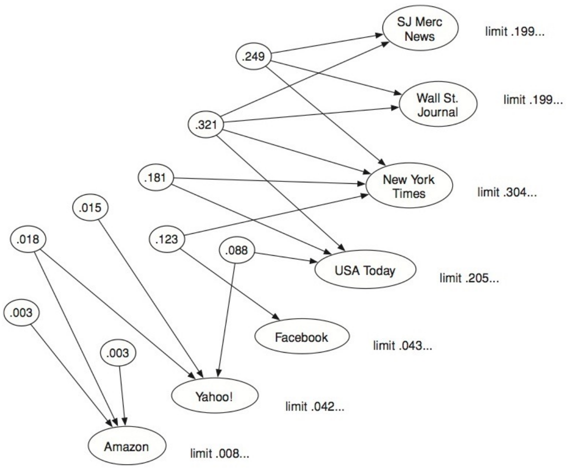


Figure 14.5. Limiting hub and authority values for the query "newspapers."

- For the "newspapers" query, our algorithm has limiting scores
- The limits are a kind of equilibrium: a balance between the authority scores (depending on the hubs pointing to them) and the hub scores (depending on the authorities they point to)
- Now that we understand the process, let's see the algorithm invented by Google, namely PageRank

PageRank

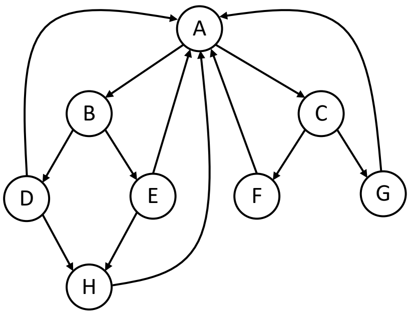
PageRank

- The PageRank algorithm was a key invention that led to the founding of Google
  - The early 1990s was the beginning of the Web, and many search engines tried to help people find good information (AltaVista, Lycos, WebCrawler, Yahoo, Ask Jeeves, etc.)
  - This continued until Google claimed most of the search engine space by introducing PageRank, which was enormously better than all the competition
- PageRank refines the hub and authority idea
  - Hubs and authorities are based on the idea that some pages are good endorsers even though they might not be endorsed themselves
  - PageRank uses only one value for a page, its importance, and assumes that if a page is important, it is also important as an endorser
  - Important pages are pointed to by important pages, and important pages point to important pages
    - A recursive definition that can be computed by an iterative algorithm

PageRank basic algorithm

- Intuitively, the PageRank value is a kind of “fluid” that circulates through the network, where nodes are containers and links are pipes
  - The amount of fluid is constant, so normalization is automatic
- Definition of PageRank algorithm (basic version):
  - Initialization: *(assume network has n nodes)*
    - $\forall p$: $pr(p) := 1/n$
  - Iteration: *(repeated k times)*
    - Compute amount of fluid on outgoing links of all pages p:
      $\forall p$: $pr_{link}(p,p') := pr(p)/m$ *(page p has m outgoing links, each to p')*
    - Sum all fluid for incoming links of page p:
      $\forall p$: $pr(p) := \sum_{p' \text{ incoming link}} pr_{link}(p',p)$

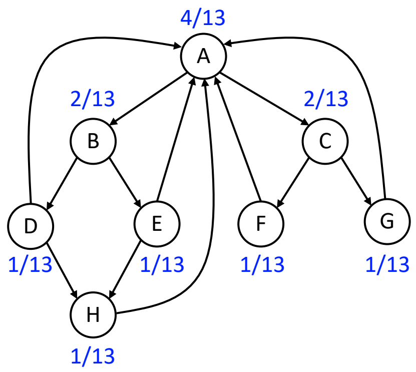
Example PageRank computation



	0	1	2	3
A	1/8			
B	1/8			
C	1/8			
D	1/8			
E	1/8			
F	1/8			
G	1/8			
H	1/8			

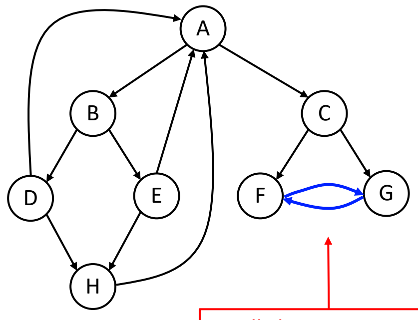
- All pages start with value 1/8
- You can work out the values at each iteration
- These values will converge to limiting values

Equilibrium values for the example



- We show the limiting values for our example
- These values are a fixpoint for the PageRank computation

Problem with the basic definition



Small change causes the basic definition to break

- There is a problem with the basic definition of PageRank
  - Some nodes can collect all the PageRank values!
- To show the problem, we change our example so that F and G point to each other
  - What is the new equilibrium value?
  - It is extreme: all the PageRank will end up at F and G!
  - F and G are 1/2, the others all 0
  - It's because there is no way out from F and G: the nodes collect everything

Fixing the problem

- We can use the “fluid” intuition to fix the problem
- Look at the water on earth: why doesn’t it all flow downward and collect at the lowest points?
  - It is because there is a counterbalancing process, namely evaporation
- To fix the PageRank definition, we extend the update rule:
  - First apply the basic update rule
  - Then do the following scaling:
    ∀p: pr(p) := s*pr(p) + (1-s)/n
  - This uses scaling factor s strictly between 0 and 1. The PageRank fraction (1-s) “evaporates” from each node and “rains” uniformly on all nodes.
  - This completes the cycle between all nodes.

The final definition of PageRank

- The final definition of PageRank uses the basic update rule (spreading of PageRank “fluid” through links) extended with the scaling rule (“evaporation” and uniform “raining” of fluid)
- We can show that this definition converges to a unique limit as the number of iterations goes to infinity
  - It’s actually an eigenvalue computation, which we will explain later
- This is the definition of PageRank used in practice, with scaling factor s chosen to be between 0.8 and 0.9
- With the scaling factor, PageRank is less sensitive to the addition of deletion of small numbers of nodes or links

Random walk definition

- We give another formulation of PageRank that is equivalent to the “fluid” formulation and that gives additional intuition
- Consider a person randomly browsing the Web
  - They start by picking a page at random (each page has equal probability)
  - They follow links for k steps and in each step they follow a random link
    - This is called a random walk on the network
  - We can prove that the probability of being on a page X after k steps is precisely the PageRank of X after k iterations of the basic update rule
- The scaled update rule is formulated as follows
  - At each step, the walker follows a random link with probability s, and “jumps” to a random page with probability (1-s)
  - The walker mostly follows the link and sometimes jumps

Modern Web search

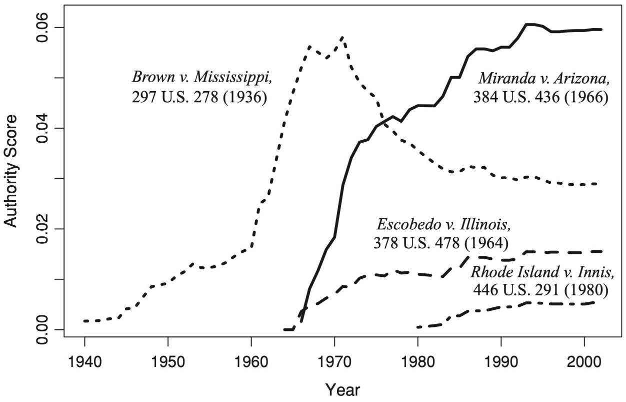
Modern Web search

- In the 1990s, the link analysis ideas (hubs and authorities, PageRank) were central to the ranking functions of Web search including Google, Yahoo!, Bing, and Ask
  - The algorithm is extended to use the anchor text, the highlighted text of a link, to describe the target page: weights are added depending on the relevance of the anchor
  - User behavior is also monitored to update the algorithm: for example, if the user often skips the first result and uses the second result, the algorithm may reorder the two first results
- However, modern search engines (since 2003) change the algorithms considerably, for two main reasons:
  1. Companies have an interest in scoring high on a ranking algorithm, which means that techniques were developed to “game” the system (cheat the PageRank value)
     - Cliff Lynch: “Web search is a new kind of information retrieval application in that the documents are actively behaving badly”
     - A large industry, Search Engine Optimization (SEO), has come into existence to do this (“link farms”) (see later!)
  2. The search industry has also added advertising to the ranking algorithms (paid results)
     - As we saw before, this uses a generalized second-price auction to determine the prices

Applications beyond the Web

- Web link analysis techniques have been used for other applications
- Importance of scientific papers
  - Garfield’s impact factor for a scientific journal (voting by in-links)
  - Pinski and Narin’s influence weights for journals (repeated improvements)
- Authority computation for US Supreme Court citations
  - Legal citations are crucial to ground decisions in precedent and explain a decision with respect to previous decisions
  - Computed authority values of decisions aligns well with qualitative judgments of legal experts [Fowler and Jeon 2008]
  - Authority of legal decisions can be computed as a function of the year, which gives a quantitative aid to legal reasoning

Authority of Fifth Amendment cases



- The US Constitutional Fifth Amendment gives an accused person certain rights: right to a grand jury, forbidding double prosecution for the same crime, and protection against self-incrimination
- Brown v. Mississippi (1936) had increasing authority, until the landmark Miranda v. Arizona (1966) case reduced the need for citing this case
- Authority computations show the evolution of the importance of these decisions in function of the year

Summary

Summary

- The Web was invented around 1990 and was instrumental in the Internet becoming a widely used information network
  - The Web is fully decentralized, with individual users both creating and accessing content
  - The Web has an emergent “bow-tie” structure with a central strongly connected component
  - The original Web has evolved to Web 2.0, with content, services, and people becoming central, instead of individual pages (increasing level of abstraction)
- Web search solves the abundance problem for information retrieval
  - No group of humans can manually classify the Web’s enormous and rapidly evolving content
  - Scalable algorithms determine relevant information using the structure of the graph itself (hubs and authorities, PageRank)
- Web search has evolved into an ecosystem that goes beyond the original PageRank
  - Search engines and companies are in a continuing struggle for ranking: search engines want to rank according to intrinsic importance, wherease companies want to rank highly
  - Search engines have added targeted advertising to the basic search functionality, turning search into a sustainable and useful business model

LINFO1115
Reasoning about
a highly connected world

Lecture 10

PageRank: random walks, matrices, and spamming

Slides by Sarunas Gurdzijauskas (KTH)
Peter Van Roy

Academic year 2022-2023
École Polytechnique de Louvain
Université catholique de Louvain

Random Walks, Matrices, and Page Rank

- This is an “invited lecture” with slides from Sarunas Gurdzijauskas
  - Slides are from courses ID2211 and ID2222 given at KTH (Stockholm, Sweden)

- This lecture continues our presentation of the PageRank algorithm
  - Random walks on graphs
    - Matrix representations for random walks
    - Eigenvalues for random walks on graphs
  - PageRank as a random walk on the Web graph
    - Fixing PageRank for the Web graph: make it strongly connected and aperiodic
  - Beyond PageRank
    - Topic-specific PageRank
    - SimRank (to find similar pages)
  - Web spamming and how to combat it
    - Link farming and how it works
    - TrustRank to measure spam mass

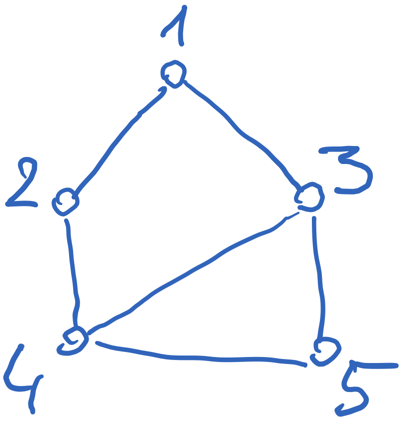
Random Walks on Graphs

Sarunas Girdzijauskas

ID2211

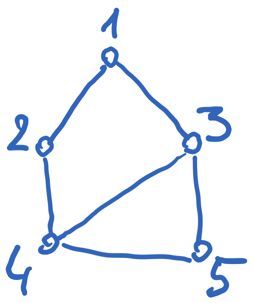
March 2019

Random Walks



- Let G(V,E) be connected graph.
- Consider random walk on G from node v.
  - We move to a neighboring node with probability 1/d(v)
  - The sequence of random walks is a Markov chain
    - A sequence of transitions according to probabilities, without memory

Random Walks



- The initial node can be fixed, but also can be drawn from some initial distribution P_0

Time	Node 1	Node 2	Node 3	Node 4	Node 5
0	1	0	0	0	0
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					

Convergence?

- For any ***connected non-bipartite*** bidirectional graph, and any starting point, the random walk converges
  - Converges to **unique** stationary distribution
  - **Power Iteration**

t	1	0	0	0	0
1	0.00	0.50	0.50	0.00	0.00
2	0.42	0.00	0.00	0.42	0.17
3	0.00	0.35	0.43	0.08	0.14
4	0.32	0.03	0.10	0.39	0.17
5	0.05	0.29	0.37	0.13	0.16
6	0.27	0.07	0.15	0.35	0.17
7	0.08	0.25	0.33	0.17	0.17
8	0.24	0.10	0.18	0.32	0.17
9	0.11	0.22	0.31	0.19	0.17
10	0.22	0.12	0.20	0.30	0.17
11	0.13	0.21	0.29	0.21	0.17
12	0.20	0.13	0.22	0.28	0.17
13	0.14	0.19	0.28	0.22	0.17
14	0.19	0.14	0.23	0.27	0.17
15	0.15	0.19	0.27	0.23	0.17
16	0.18	0.15	0.23	0.27	0.17
17	0.15	0.18	0.26	0.24	0.17
18	0.18	0.16	0.24	0.26	0.17
19	0.16	0.18	0.26	0.24	0.17
20	0.17	0.16	0.24	0.26	0.17
21	0.16	0.17	0.26	0.24	0.17
22	0.17	0.16	0.24	0.26	0.17
23	0.16	0.17	0.25	0.25	0.17
24	0.17	0.16	0.25	0.25	0.17
25	0.16	0.17	0.25	0.25	0.17
26	0.17	0.16	0.25	0.25	0.17
27	0.16	0.17	0.25	0.25	0.17
28	0.17	0.16	0.25	0.25	0.17
29	0.17	0.17	0.25	0.25	0.17
30	0.17	0.17	0.25	0.25	0.17

Stationary Distribution

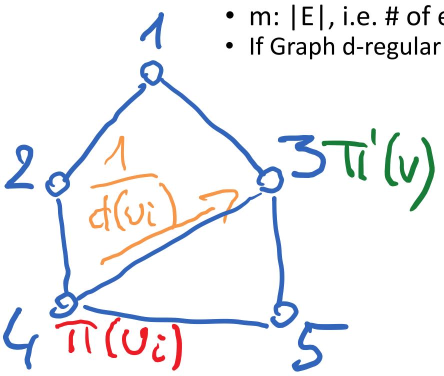
- To which distribution does the random walk converge in our graph?

π	0.17	0.17	0.25	0.25	0.17
π'	0.17	0.17	0.25	0.25	0.17

- Random walk converges to the stationary distribution:

$\pi(v) = d(v)/2m$

  - d(v) = degree of v, i.e. # of neighbors.
  - m: |E|, i.e. # of edges.
  - If Graph d-regular then to uniform distribution



$$\pi'(v) = \sum_{u:\,(u,v)\in E} \pi(u) \frac{1}{d(u)}$$

$$= \sum_{u:\,(u,v)\in E} \frac{d(u)}{2m} \cdot \frac{1}{d(u)}$$

$$= \sum_{u:\,(u,v)\in E} \frac{1}{2m}$$

$$= \frac{d(v)}{2m}$$

$$= \pi(v)$$

Implications

- The stationary distribution

  $\pi(v) = d(v)/2m$

  is **proportional to the degree of v**.
  - What's the intuition?
  - The more neighbors you have, the more chance you'll be visited.

Matrix Representations

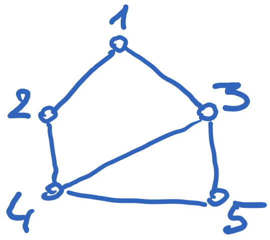
Definitions

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Adjacency Matrix

$$D = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

Diagonal matrix with $D_{i,i} = 1/d(i)$



$$M= \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

Transition (random walk) Matrix
M=DA

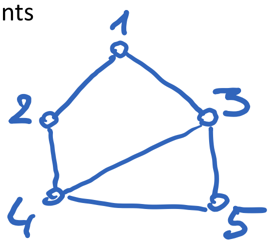
Adjacency Matrix

- A is *n x n* adjacency matrix of G=(V,E)
  - A_{ij} is 1 if there is a link between i and j nodes and 0 otherwise
- Gives us all 1-hop paths.
- How to count # of 2-hop paths?
  - A^2

Matrix manipulations

- A^2 gives us # of 2-hop paths
- A^3 gives us ?
  - # of 3-hop paths, etc.
    - Technically it is not a path, but a walk (not simple paths, because you can visit the same node several times)
- What about taking a vector v=(1 0 0 0 0) that represents a message at the first node and multiplying it by A?

$$vA = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$



- vA=(0 1 1 0 0)
  - indicates how many walks of length 1 from node 1 end up in node i.
- vA^2 = (2 0 0 2 1)
  - Indicates how many walks of length 2 from node 1 end up in node i.
- vA^3 = (0 4 5 1 2)
  - Indicates how many walks of length 3 from node 1 end up in node i.

Matrix manipulations (cont.)

- What about multiplying by a Random Walk Matrix?

$$M=\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

Transition (random walk) Matrix
M=DA

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}=\begin{pmatrix} 0.42 & 0 & 0 & 0.42 & 0.17 \end{pmatrix}$$

What about Random Walk Matrix M? (cont.)

- Recap: Random walk on a graph G: we start at a node v_0 and at the t-th step we are at a node v_t. We move to a neighbor of v_t with probability $1/d(v_t)$.
  - The sequence of random nodes (v_t :t=0,1,2…) is a Markov chain
- We start from the initial state of the system, e.g. P_0: [1 0 0 0 0];
  - Can also be drawn from some initial distribution
- **$P_t = P_0M^t$**
  - Or can be written as $P_t = (M^T)^tP_0$ if we represent P as a column vector

Again the same Example

$P_t = P_0 M^t$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} 0.42 & 0 & 0 & 0.42 & 0.17 \end{pmatrix}$$

- When $P_{t+1} = P_t = \pi$, we have reached stationary distribution, i.e. **$\pi M = \pi$**
- Recall: that v is **eigenvector** of matrix M and λ its eigenvalue if **$vM = \lambda v$**
  - so ***π is eigenvector of M with eigenvalue $\lambda=1$***

Random Walks and Eigenvalues

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March 2019

Random Walk

$P_t = P_0 M^t$

$$(1 \quad 0 \quad 0 \quad 0 \quad 0)\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix} = (0 \quad 1/2 \quad 1/2 \quad 0 \quad 0)$$

$$(0 \quad 1/2 \quad 1/2 \quad 0 \quad 0)\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix} = (0.42 \quad 0 \quad 0 \quad 0.42 \quad 0.17)$$

- When $P_{t+1} = P_t = \pi$, we have reached stationary distribution, i.e. **$\pi M = \pi$**
- Recall: that v is **eigenvector** of matrix M and λ its eigenvalue if **$vM = \lambda v$**
  - so ***π is eigenvector of M with eigenvalue $\lambda=1$***

Intuition on spectra of matrix A

- First example: d-regular graph (all vertices degree d, connected & unweighted for now)
- Recap: What is the meaning of *Ax* ?

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Consider **x** as a vector representing values for each node in the graph

$$y_j = \sum_{i=1}^{n} A_{ij} x_i = \sum_{(j,i)\in E} x_i$$

Entry y_j is a sum of labels x_i of neighbors of i

- And what is an eigenvector of *A* ?

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boldsymbol{A \cdot x = \lambda \cdot x}$$

Set of eigenvalues and eigenvectors

Finding principal eigenvector of A

- Suppose all nodes in $\boldsymbol{G}$ have degree $\boldsymbol{d}$ and $\boldsymbol{G}$ is connected (i.e., G is a d-regular graph)
- **What are some eigenvalues/vectors of $\boldsymbol{G}$? $\boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{\lambda}\cdot\boldsymbol{x}$**
  What is $\boldsymbol{\lambda}$? What $\boldsymbol{x}$?
  - **Let's try:** $\boldsymbol{x} = (\mathbf{1}, \mathbf{1}, \dots, \mathbf{1})$
  - **Then:** $\boldsymbol{A}\cdot\boldsymbol{x} = (\boldsymbol{d}, \boldsymbol{d}, \dots, \boldsymbol{d}) = \boldsymbol{\lambda}\cdot\boldsymbol{x}$. **So:** $\boldsymbol{\lambda} = \boldsymbol{d}$
  - **We found eigenpair of $\boldsymbol{G}$: $\boldsymbol{x} = (\mathbf{1}, \mathbf{1}, \dots, \mathbf{1})$, $\boldsymbol{\lambda} = \boldsymbol{d}$**
- This is the principal eigenvector of A, $\boldsymbol{\lambda}$=d
- But for non d-regular graphs the eigenvectors are much more interesting!

Meaning: Each node asks for the labels of each neighbor and sums them up

Remember the meaning of $y = A\cdot x$:

$$y_j = \sum_{i=1}^{n} A_{ij} x_i = \sum_{(j,i)\in E} x_i$$

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Graph Spectra (Set of Eigenvalues)

- **vA=λv** (Av= λv for column vector)
- If A is a real symmetric matrix then it has n eigenvectors and associated n eigenvalues. All n eigenvalues are real $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, and the eigenvectors are orthogonal
  - We saw that if G is a d-regular graph then λ_1=d
- For random walk matrix M, i.e., normalized adjacency matrix λ_1=1 **(πM= π)**

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Graph Laplacian (for your information…)

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Adjacency Matrix

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

Laplacian Matrix

Degree matrix: diagonal matrix where each entry gives the degree of the vertex

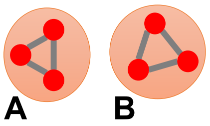
- **Laplacian Matrix L=D-A** where **D** is a degree matrix
  - λ_1 =0
- If $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ eigenvalues of L then:
  - G has k connected components if λ_k =0
- We mention the Laplacian here because it is important in the theory of graphs, but we will not use it further in the lecture

Graph Spectra (cont.)

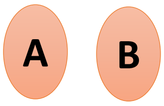
- We call λ_1, λ_2,…, λ_n the spectrum of graph G
  - from matrices A, M or L
- We call λ_1-λ_2 the eigengap (or **spectral gap**)
  - 1-λ_2 for M
- So what if the graph is disconnected?
  - Think of convergence of Random Walk on M…
  - Very interesting consequence: λ_1=λ_2
  - Let us see why it is so…

Intuition: Disconnected Graph

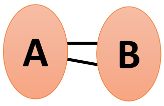
- **What if G is not connected?**
  - G has $\mathbf{2}$ components, each d-regular



- **What are some eigenvectors?**
  - $x =$ Put all $\mathbf{1}$s on A and $\mathbf{0}$s on B or vice versa
    - $x' = (\underbrace{1, \dots, 1}_{|A|}, 0, \dots, 0)$ **then** $A \cdot x' = (d, \dots, d, 0, \dots, 0)$
    - $x'' = (0, \dots, 0, \overset{|B|}{\overline{1, \dots, 1}})$ **then** $A \cdot x'' = (0, \dots, 0, d, \dots, d)$
    - And so in both cases the corresponding $\lambda = d$
- **Further intuition:**



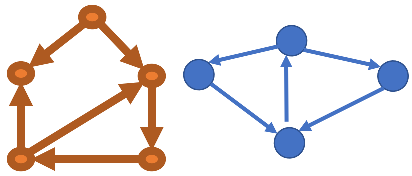
$\lambda_1 = \lambda_2$



$\lambda_1 - \lambda_2 \approx 0$

Convergence of Random Walk

- For every ***connected non-bipartite*** undirected graph G, the distribution Pi converges to a limit and unique stationary distribution **π**.
- Moreover, if G is regular then this distribution is the uniform distribution on V.
- Intuition:
  - Why connected?
  - Why non-bipartite?
- What about directed graphs?



  - Has to be ***strongly connected***! (otherwise the "walks will leak").
  - Has to be ***aperiodic,*** i.e., visits to some state (node) S should never be a multiple of k (k>1)
    - if the greatest common divisor of the lengths of all its cycles is 1

Google PageRank

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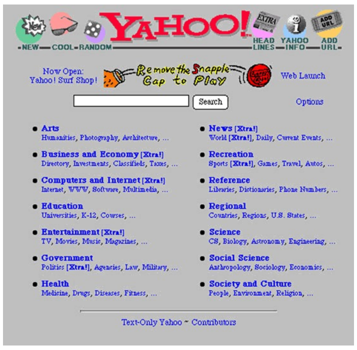
March 2019

History of Web search

- **1st generation (Directories):**
  - Manual curation of **web directories** (e.g., Yahoo)
  - Web was growing too quickly to catch up
- **2nd generation (Information Retrieval)**
  - – Altavista
  - – Classical Information retrieval, processing text in the pages
  - – Term Spam
- **3rd generation (Google)**
  - – Google PageRank
    - Very hard to fake in-links.



Now Open: Yahoo! Surf Shop!

Web Launch

Options

- **Arts**
  Humanities, Photography, Architecture, ...
- **Business and Economy** [Xtra!]
  Directory, Investments, Classifieds, Taxes, ...
- **Computers and Internet** [Xtra!]
  Internet, WWW, Software, Multimedia, ...
- **Education**
  Universities, K-12, Courses, ...
- **Entertainment** [Xtra!]
  TV, Movies, Music, Magazines, ...
- **Government**
  Politics [Xtra!], Agencies, Law, Military, ...
- **Health**
  Medicine, Drugs, Diseases, Fitness, ...
- **News** [Xtra!]
  World [Xtra!], Daily, Current Events, ...
- **Recreation**
  Sports [Xtra!], Games, Travel, Autos, ...
- **Reference**
  Libraries, Dictionaries, Phone Numbers, ...
- **Regional**
  Countries, Regions, U.S. States, ...
- **Science**
  CS, Biology, Astronomy, Engineering, ...
- **Social Science**
  Anthropology, Sociology, Economics, ...
- **Society and Culture**
  People, Environment, Religion, ...

Text-Only Yahoo ~ Contributors

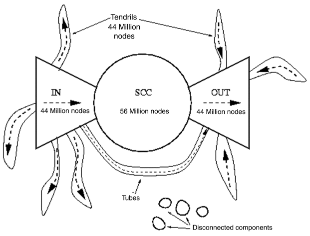
PageRank ”Voting” formulation

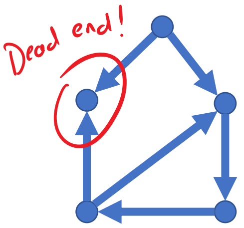
- Each page has a budget of ”votes” and distributes them evenly to all the outgoing links
    - E.g., if page j has r_j budget of votes, and n out-links, each link gets r_j/n votes.
    - INSIGHT: **A vote from an important page is worth more.**
- Node’s j own importance is the **sum of the votes on its in-links**.
- Did we see this before??
    - Notice the similarity with the **random walk**

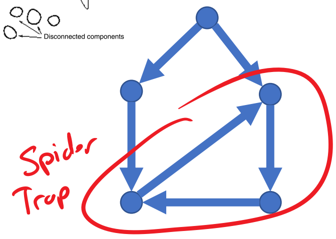
Google PageRank

Does this Graph have a unique stationary distribution of the random walk?

- Google page rank: **principal eigenvector** on the transition matrix of the web graph!
    - How to compute?
        - Power iteration.
    - Any issues?
        - Undirectional vs directional graph?
            - **Dead ends**
                - Nodes with no out-degree
                - Votes leak out
            - **Spider traps**
- WWW is not strongly connected!




Dead end!

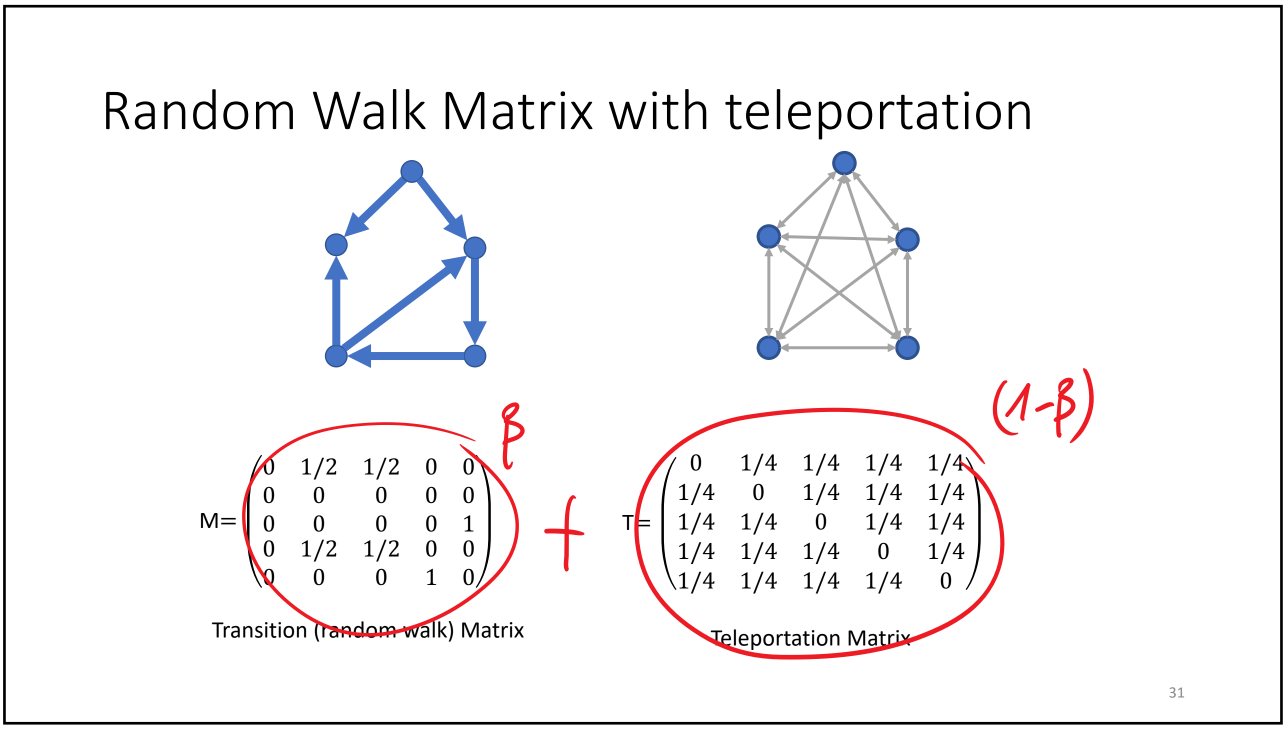

Spider Trap

How do we fix PageRank?

- Ideas?
  - Make the graph **strongly connected** and **aperiodic**
  - Google solution
    - Make ”**tiny tiny**” links from each node to every other node
    - Keep **the core** of the initial graph

How do we fix PageRank? (cont.)

- Google random walker will
  - With **prob β** follow ”the real” link at random.
  - With **prob 1-β** jump to some random page.
  - Usually β is in the range of 0.8 to 0.9
    - I.e., random walker will ”teleport” from any spider trap after 5-10 steps
  - For **”dead ends”** **1-β=1**, i.e., once in the dead end, random walker always teleports to a random node.



Random Walk Matrix with teleportation

$$\beta \qquad (1-\beta)$$

$$M=\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} + \quad T=\begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix}$$

Transition (random walk) Matrix

Teleportation Matrix

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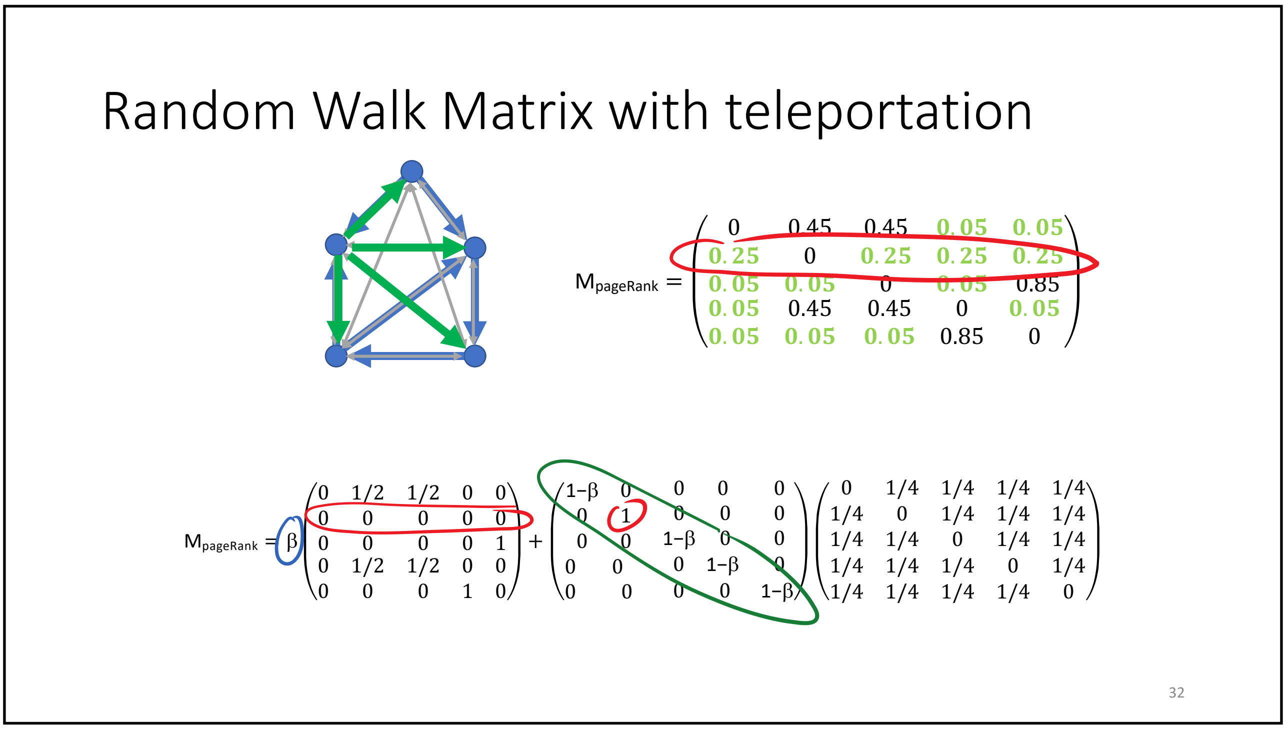

Random Walk Matrix with teleportation

$$M_{pageRank} = \begin{pmatrix} 0 & 0.45 & 0.45 & 0.05 & 0.05 \\ 0.25 & 0 & 0.25 & 0.25 & 0.25 \\ 0.05 & 0.05 & 0 & 0.05 & 0.85 \\ 0.05 & 0.45 & 0.45 & 0 & 0.05 \\ 0.05 & 0.05 & 0.05 & 0.85 & 0 \end{pmatrix}$$

$$M_{pageRank} = \beta \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1-\beta & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-\beta & 0 & 0 \\ 0 & 0 & 0 & 1-\beta & 0 \\ 0 & 0 & 0 & 0 & 1-\beta \end{pmatrix} \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix}$$

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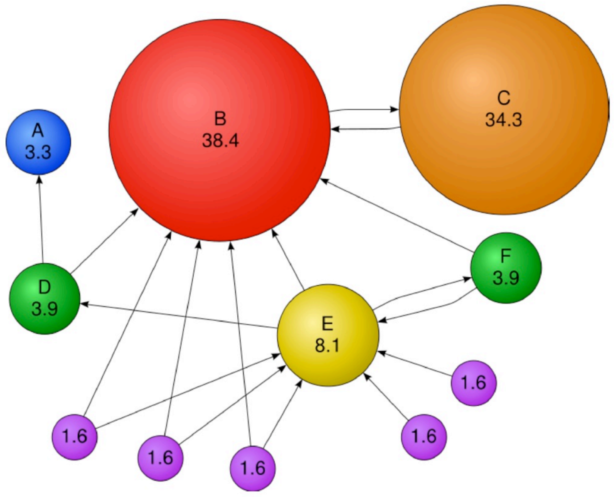
Is it fixed now?

- What if **β=0** for all the nodes?
- What are the problems with teleportation?
  - How does $M_{pageRank}$ look for 10bn pages?
    - Dense random walk matrix!
    - N^2 non-zero elements! (instead of O(N*d))
  - Can you even store it in memory?
  - Insight for the Fix:
    - Interpret teleportation as **fixed tax** (always the same),
    - At every power iteration instead of computing rank vector $\mathbf{r}^{new}$
      $\mathbf{r}^{new}=\mathbf{r}^{old}\, M_{pageRank}$, we compute $\mathbf{r}^{new}= \beta\ (\mathbf{r}^{old} M)+c$,
      where $c = (1- \beta)/N$ (i.e., a tax)
    - Notice $M_{pageRank}$ is dense and M is sparse matrix!
    - If M contains dead-ends then r^{new} has to be made stochastic again i.e., so that it sums up to 1.

Example

Node size proportional to the PageRank score



Problems with PageRank?

- **Measures ”generic popularity” of a page**
  - Might miss topic-specific authorities
  - We solve it by **Topic-Specific** PageRank
- **Susceptible to link spam**
  - Artificial link topologies created to boost page rank
  - We solve it by **TrustRank**
- **Uses a single measure of importance**
  - We can address it by **Hubs-and-Authorities**
  - This was seen in LINFO1115 in Lecture 9

Beyond PageRank

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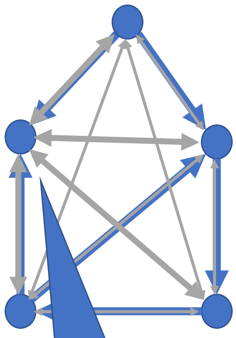
March 2019

Topic Specific PageRank

- Insight: **Bias the random walk** towards ”relevant set nodes”
  - Give **more influence** to the webpages that are close to a particular topic (given by a query), e.g., ”sports”, ”travel” etc
- Instead of teleporting to ”any node” - teleport to ”relevant pages” (**teleport set**) for topic-specific PageRank
  - Could also assign different weights to pages within the teleport set
- Once we have a biased (green) weights we recalculate PageRank in a regular fashion

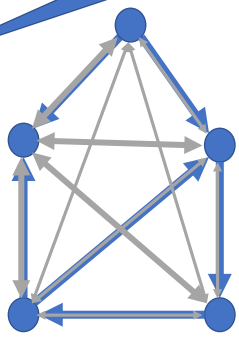


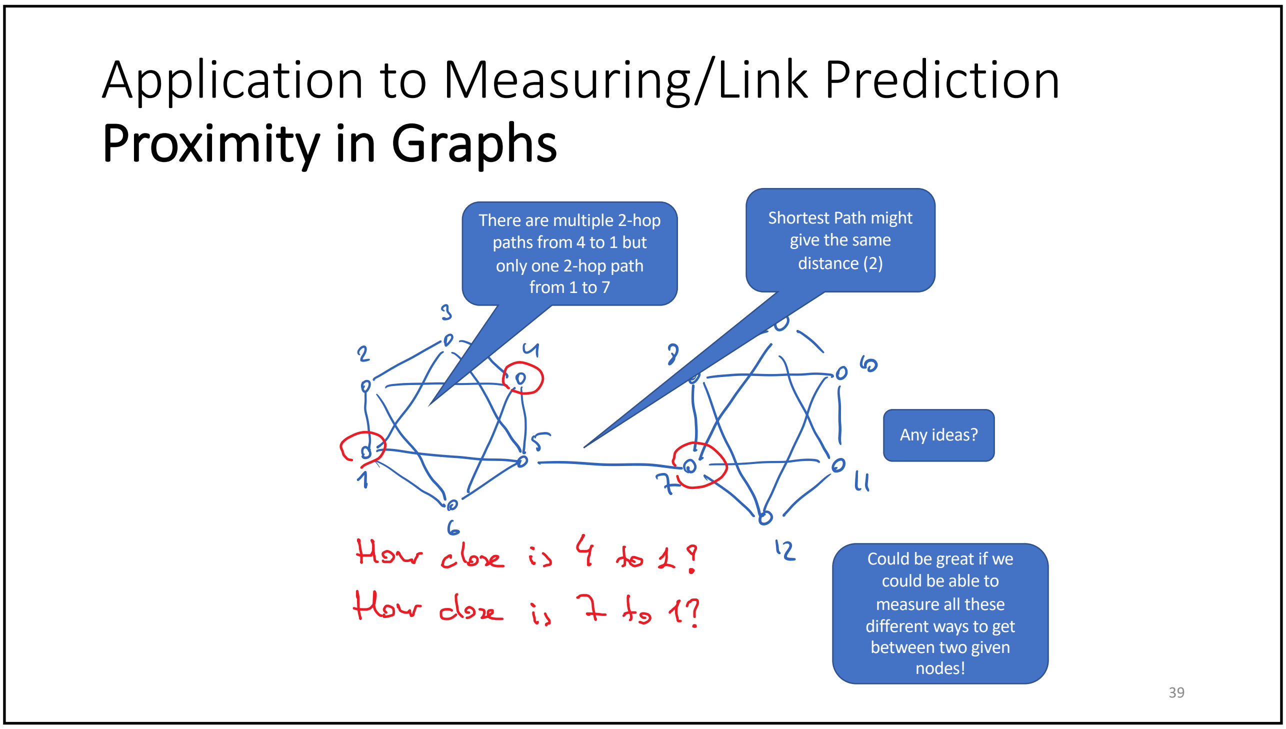
We can move all the extra weight to the teleportation set

Topic Specific PageRank (cont.)

Take the sets from the ”previous generation” search engines

- One can precalculate PageRanks for each page for different topics
  - E.g., arts, business, sports etc.
- How do we get the **teleport set?**
- Which topic ranking to use?
  - User can pick from a menu
  - Classify query into a topic
  - Exploit the **”context”** of the query:
    - Where the query is launched from (a topic specific webpage?)
    - User browsing history (e.g., could be a difference if one queries “Manchester” after “football” or after “travel” queries)
    - User bookmarks, cookies etc.





Application to Measuring/Link Prediction
Proximity in Graphs

There are multiple 2-hop paths from 4 to 1 but only one 2-hop path from 1 to 7

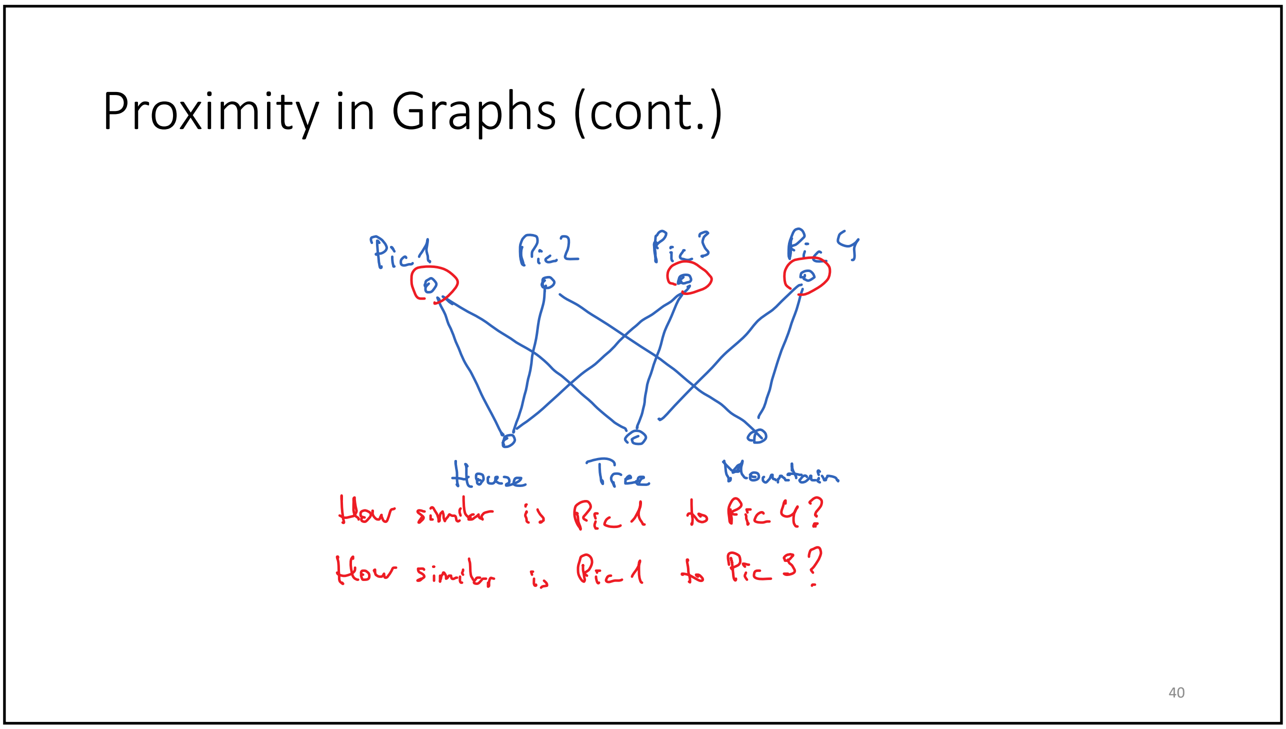
Shortest Path might give the same distance (2)

Any ideas?

How close is 4 to 1?

How close is 7 to 1?

Could be great if we could be able to measure all these different ways to get between two given nodes!



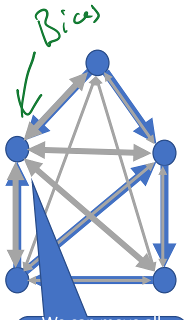
Proximity in Graphs (cont.)

How similar is Pic 1 to Pic 4?

How similar is Pic 1 to Pic 3?

Topic Specific PageRank

- Insight: **Bias the random walk** towards ”relevant set nodes”
- Instead of teleporting to ”any node” - teleport to ”relevant pages” (**teleport set**) for topic-specific PageRank
- Once we have a biased (green) weights we recalculate PageRank in a regular fashion
- **What happens if our teleport set is the “initial node itself”?**



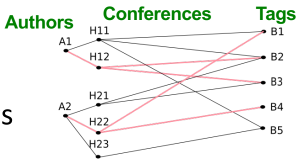
We can move all the extra weight to the teleportation set

SimRank: Idea

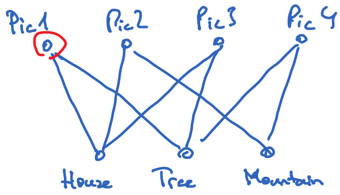
G.Jeh et al. SimRank: A Measure of Structural-Context Similarity

AKA PageRank with Restarts

- **Topic Specific PageRank** from node ***u***: **teleport set *S* = {*u*}**
  - Resulting scores measures similarity to node ***u***
- **SimRank:** Random walks from a **fixed node** on ***k***-partite graphs
- **Setting:** ***k***-partite graph with ***k*** types of nodes
  - E.g.: Authors, Conferences, Tags
- **Any problems?**
  - Must be done once for each node ***u***
  - Used in *Pinterest* for recommendation.



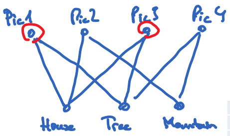
SimRank Example



- What is the most related picture to Pic1?
- Topic-Specific PageRank with teleport set ={Pic1}
    - Random Walk with restart

SimRank Example (cont.)

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.3333 & 0.3333 & 0 \\ 0 & 0 & 0 & 0 & 0.3333 & 0 & 0.5000 \\ 0 & 0 & 0 & 0 & 0.3333 & 0.3333 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.3333 & 0.5000 \\ 0.5000 & 0.5000 & 0.5000 & 0 & 0 & 0 & 0 \\ 0.5000 & 0 & 0.5000 & 0.5000 & 0 & 0 & 0 \\ 0 & 0.5000 & 0 & 0.5000 & 0 & 0 & 0 \end{pmatrix}$$



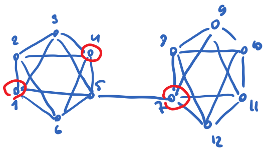
1	**0.7287**
2	0.3179
3	0.3102
4	**0.2404**
5	0.3298
6	0.3102
7	**0.0744**
8	0.0271
9	0.0271
10	0.0216
11	0.0271

Restart on Pic1 ($\beta = 0.8$)

$v =$

Pic1	**0.3024**
Pic2	0.0753
Pic3	**0.1024**
Pic4	0.0753
House	0.1921
Tree	0.1921
Mountain	0.0603

$$M_R = \begin{pmatrix} 0.2000 & 0.2000 & 0.2000 & 0.2000 & 0.4667 & 0.4667 & 0.2000 \\ 0 & 0 & 0 & 0 & 0.2667 & 0 & 0.4000 \\ 0 & 0 & 0 & 0 & 0.2667 & 0.2667 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2667 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0 & 0 & 0 & 0 \\ 0.4000 & 0 & 0.4000 & 0.4000 & 0 & 0 & 0 \\ 0 & 0.4000 & 0 & 0.4000 & 0 & 0 & 0 \end{pmatrix}$$



Calculate $My = v$

PageRank: Summary

- **“Normal” PageRank:**
  - Teleports uniformly at random to any node
  - All nodes have the same probability of surfer landing there
- **Topic-Specific PageRank also known as Personalized PageRank:**
  - Teleports to a topic specific set of pages
  - Nodes can have different probabilities of surfer landing there
- **Random Walk with Restarts (SimRank):**
  - Topic-Specific PageRank where teleport is always to the same node
  - Useful to find nodes that are similar to a given node

Web spamming and how to combat it

Web Spam

- **What is web – spamming?**
  - A deliberate action to boost a webpage position in ranking.
  - *~10-15% pages are spam?*
    - *I.e., pages appearing in your rank results that should not be there*
    - *Some people say it is just SEO (Search Engine Optimization)*

Web search

- **Early search engines:**
  - Crawl the Web
  - Index pages by the words they contained
  - Respond to search queries (lists of words) with the pages containing those words
- **Early page ranking:**
  - Attempt to order pages matching a search query by "importance"
  - **First search engines considered:**
    - **(1)** Number of times query words appeared
    - **(2)** Prominence of word position, e.g. title, header

First Spammers

- As people began to use search engines to find things on the Web, those with commercial interests tried to **exploit search engines** to bring people to their own site – whether they wanted to be there or not
- **Example:**
  - Shirt-seller might pretend to be about “movies”
- **SPAM - Techniques for achieving high relevance/importance for a web page**

First Spammers: Term Spam

- **How do you make your page appear to be about movies?**
  - **(1)** Add the word movie 1,000 times to your page
  - Set text color to the background color, so only search engines would see it
  - **(2)** Or, run the query “movie” on your target search engine
  - See what page came first in the listings
  - Copy it into your page, make it “invisible”
- **These and similar techniques are called “term spam”**

Google’s Solution to Term Spam

- **Believe what people say about you, rather than what you say about yourself**
  - Use words in the anchor text (words that appear underlined to represent the link) and its surrounding text
- PageRank as a tool to measure the “importance” of Web pages

Why It Works?

- **Our hypothetical shirt-seller loses. Why?**
  - Saying he is about movies doesn’t help, because others don’t say he is about movies
  - His page isn’t very important, so it won’t be ranked high for shirts or movies
- **What can you do to game it?**
  - Shirt-seller creates 1,000 pages, each links to his with “movie” in the anchor text
  - BUT, these pages have no in-links, so they get little PageRank?
    - So the shirt-seller can’t beat truly important movie pages, like IMDB
- **Unless… it is a coordinated attack or Spam Farming**

Coordinated Attack

Web Results **1** - **10** of about **969,000** for **miserable failure**. (**0.06** seconds)

Biography of President George W. Bush
Biography of the president from the official White House web site.
www.whitehouse.gov/president/gwbbio.html - 29k - Cached - Similar pages
Past Presidents - Kids Only - Current News - President
More results from www.whitehouse.gov »

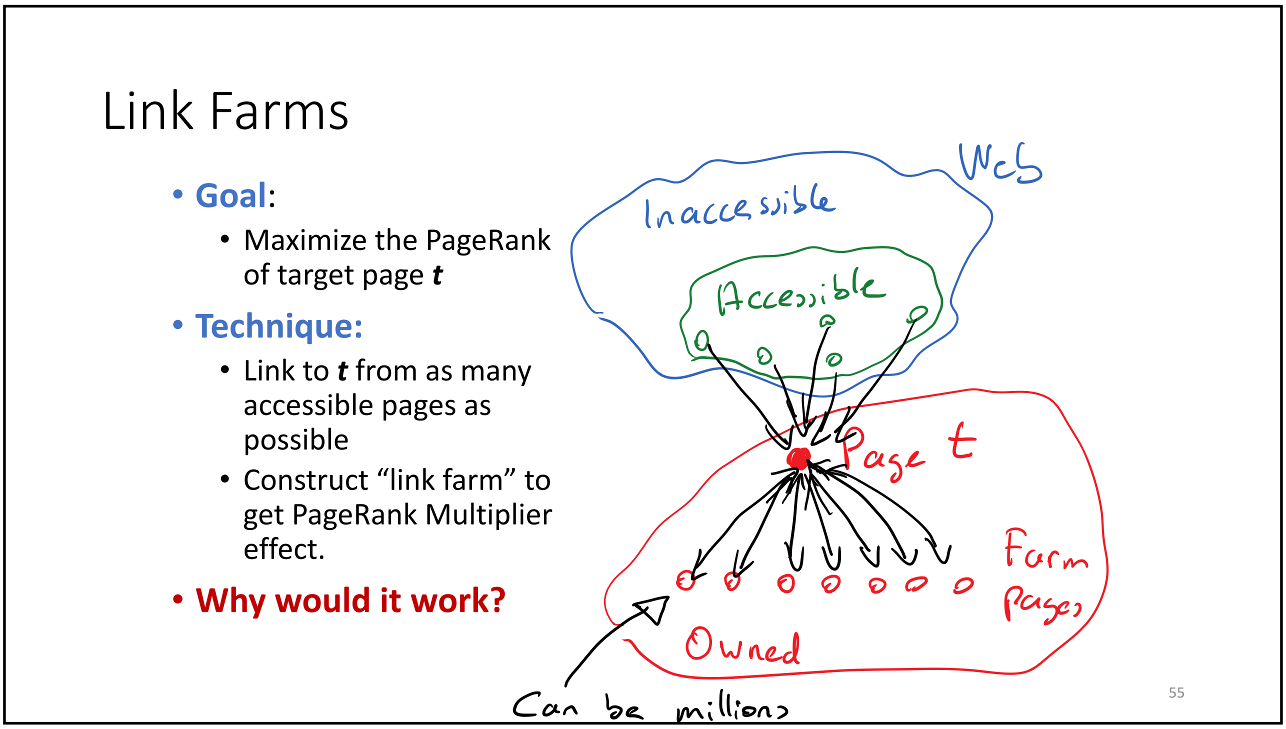
Welcome to MichaelMoore.com!
Official site of the gadfly of corporations, creator of the film Roger and Me and the television show The Awful Truth. Includes mailing list, message board, ...
www.michaelmoore.com/ - 35k - Sep 1, 2005 - Cached - Similar pages

BBC NEWS | Americas | '**Miserable failure**' links to Bush
Web users manipulate a popular search engine so an unflattering description leads to the president's page.
news.bbc.co.uk/2/hi/americas/3298443.stm - 31k - Cached - Similar pages

Google's (and Inktomi's) **Miserable Failure**
A search for **miserable failure** on Google brings up the official George W. Bush biography from the US White House web site. Dismissed by Google as not a ...
searchenginewatch.com/sereport/article.php/3296101 - 45k - Sep 1, 2005 - Cached - Similar pages

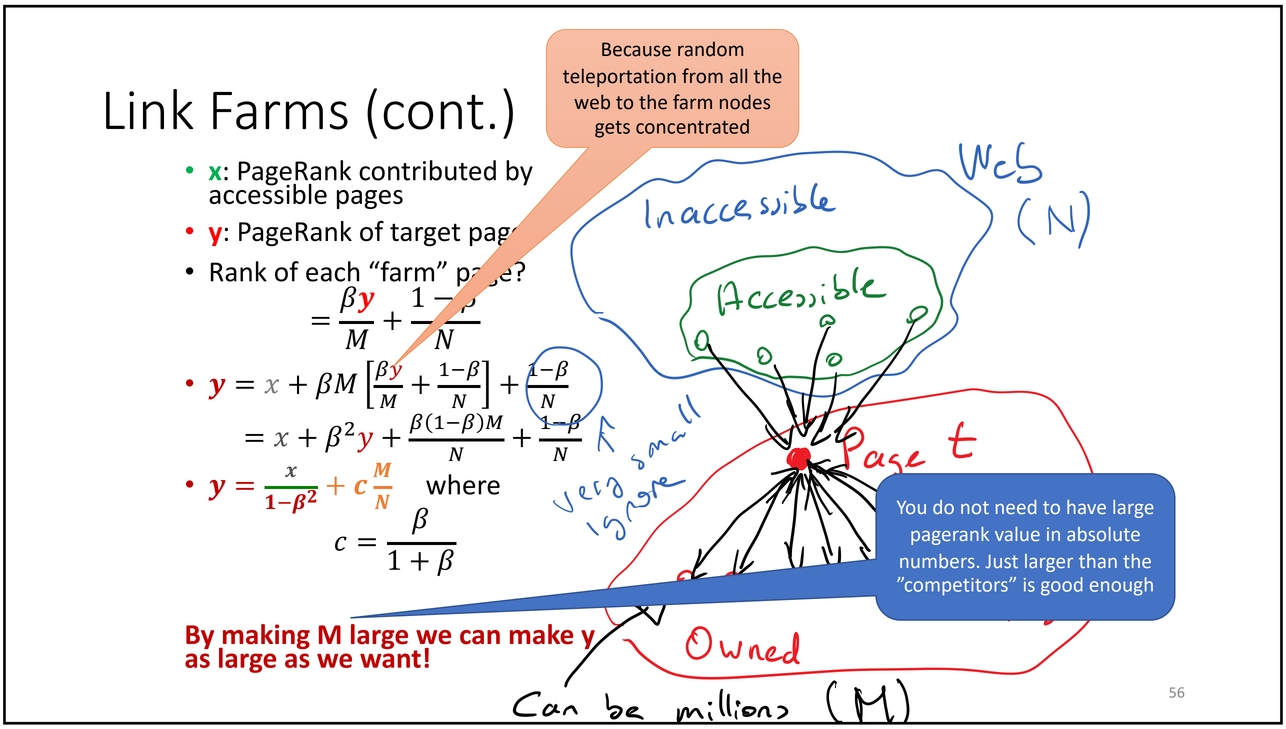
Spam Farming

- **Concentrating PageRank** on a single page.
  - Creating Link structures that boost Page rank of a particular page
- Webpages from spammers point of view:
  - **Inaccessible** pages
  - **Accessible** pages
    - Blogs, comments, reviews, where spammer can post.
  - **Owned** pages
    - Completely controlled by spammer
    - Could be from multiple domains



Link Farms

- **Goal**:
  - Maximize the PageRank of target page ***t***
- **Technique:**
  - Link to ***t*** from as many accessible pages as possible
  - Construct “link farm” to get PageRank Multiplier effect.
- **Why would it work?**



Link Farms (cont.)

- **x**: PageRank contributed by accessible pages
- **y**: PageRank of target page
- Rank of each “farm” page?

$$= \frac{\beta \boldsymbol{y}}{M} + \frac{1-\beta}{N}$$

- $\boldsymbol{y} = x + \beta M \left[\frac{\beta y}{M} + \frac{1-\beta}{N}\right] + \frac{1-\beta}{N}$

$$= x + \beta^2 y + \frac{\beta(1-\beta)M}{N} + \frac{1-\beta}{N}$$

- $\boldsymbol{y} = \frac{\boldsymbol{x}}{\boldsymbol{1-\beta^2}} + \boldsymbol{c}\frac{\boldsymbol{M}}{\boldsymbol{N}}$ where

$$c = \frac{\beta}{1+\beta}$$

By making M large we can make y as large as we want!

Because random teleportation from all the web to the farm nodes gets concentrated

You do not need to have large pagerank value in absolute numbers. Just larger than the ”competitors” is good enough

Topic Specific PageRank

- How to **combat LinkSpam**?
  - Ideas?
  - **Identifying and Blacklisting of Spam farms?**
    - Arms race of hiding and detecting spam farms
  - **TrustRank** - teleport to trusted pages only
    - University homepages, trusted domains (.edu) etc
    - Manually preselected “good” pages.
    - Idea behind: it is rare for a “good” page to point to a “bad” (spam) page
  - **Teleport set = trusted pages**
    - Similar to Topic-specific PageRank
    - Effectively propagating trust through links (e.g., each page gets a trust value between 0 and 1)
      - Similar to the proximity calculation
      - Trust is **split** across the out-links (the larger the number of out-links, the less scrutiny the page author gives to each outlink)
      - The **degree of trust** conferred by a trusted page **decreases with the distance** in the graph

How to pick Trusted Seed Set?

- **Two conflicting considerations:**
  - Human has to inspect each seed page, so seed set must be **as small as possible**
  - But we also would like to our set to be **as big as possible**, ideally including every **good page** on the web
- Picking k seed pages
  - 1. **PageRank itself** – pick the top k pages produced by regular RageRank (assumption is that you can’t get a bad page’s rank really high)
  - 2. **Trusted domains** whose membership is controlled (.edu,.mil,.gov etc)

Topic Specific Page Rank

- How to **combat LinkSpam**?
  - Ideas?
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    - War of hiding and detecting spam farms
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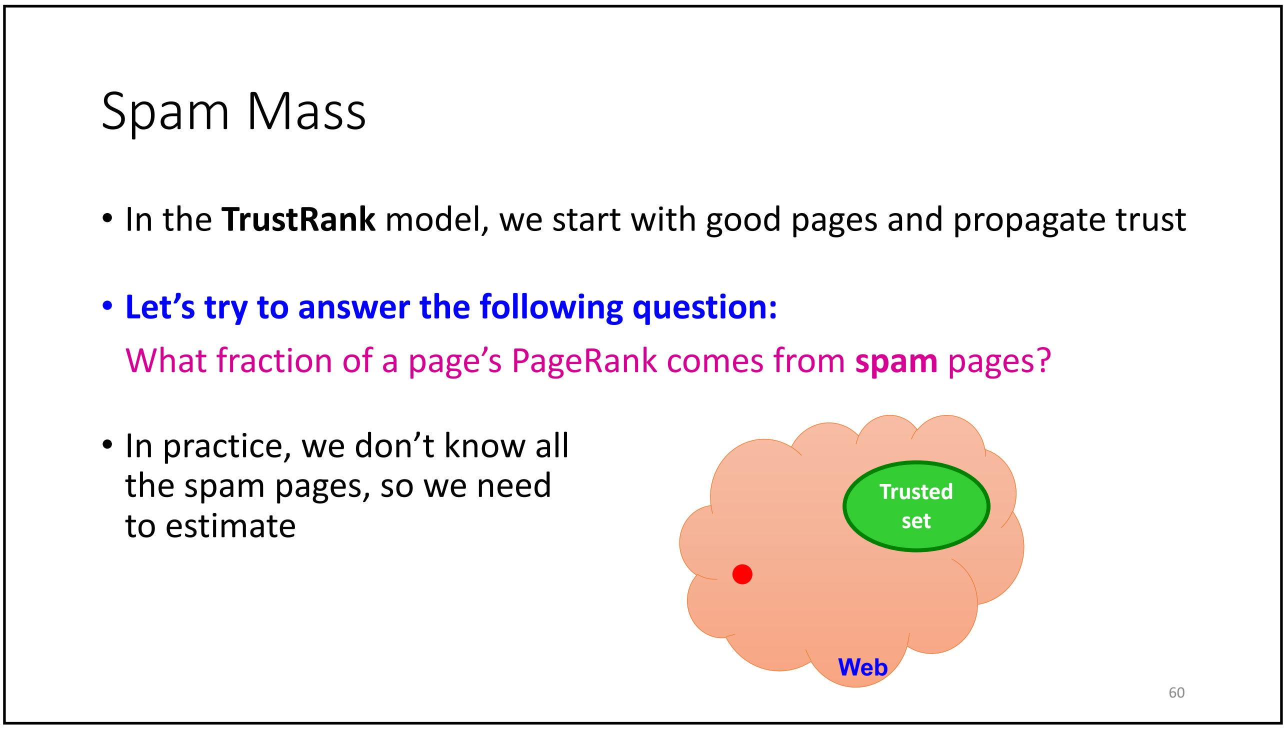
      - Trust is **split** across the out-links (the larger the number of out-links, the less scrutiny the page author gives to each outlink)
      - The **degree of trust** conferred by a trusted page **decreases with the distance** in the graph
    - **Solution 1: Mark as spam** all pages **below the trust threshold**.
    - Problems?
      - Some pages that were legitimate but naturally have low page rank (e.g., new pages) will be ignored
      - This is because we look at the **absolute value** of the page rank. What else could we do?



Spam Mass

- In the **TrustRank** model, we start with good pages and propagate trust

- **Let’s try to answer the following question:**

  What fraction of a page’s PageRank comes from **spam** pages?

- In practice, we don’t know all the spam pages, so we need to estimate

Spam Mass Estimation

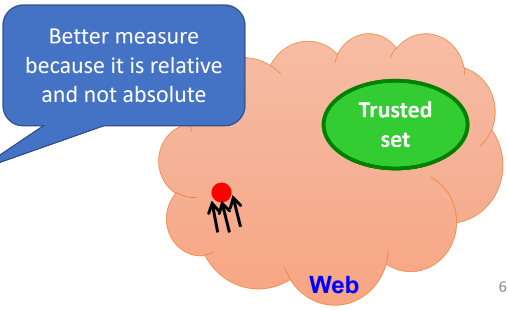
Solution 2:

- $\boldsymbol{r_p}$ = PageRank of page ***p***
- $\boldsymbol{r_p^+}$ = PageRank of ***p*** with teleport into **trusted** pages only

- **Then:** What fraction of a page's PageRank comes from **spam** pages?

$$\boldsymbol{r_p^- = r_p - r_p^+}$$

- **Spam mass of *p* =** $\boldsymbol{\frac{r_p^-}{r_p}}$
  - Pages with high spam mass are spam.



Better measure because it is relative and not absolute

Summary

Deeper study of PageRank

- PageRank is just one of many powerful techniques for studying the Web
  - PageRank can be seen as a form of random walk
  - We study random walks and their convergence properties
- Random walks can be represented through matrix representations
  - Eigenvalues and eigenvectors are powerful tools for understanding
- PageRank history and definition
  - History of Web search and how it led to PageRank
- Problems with PageRank and solutions
  - The matrices are huge: how do we compute the solution?
  - How do we target PageRank to specific topics?
    - Topic-specific PageRank (teleport to specific topics)
    - SimRank (teleport to same node, useful for finding similar nodes)
  - What is Web spamming and how can we combat it?
    - Link farms can increase PageRank, use TrustRank to estimate spam mass

LINFO1115
Reasoning about a highly connected world

Lecture 11
Information cascades

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Network dynamics

- Network dynamics focuses on how networks influence group behavior
- Cascades
  - Information cascades (ch. 16) ← This lecture
    - Sequential decision making versus independent decision making
    - Bayes' Rule and an idealized cascade model explain how cascades form
  - Direct-benefit cascades (also called network effects) (ch. 17)
    - A second kind of cascade exists when you get a benefit by aligning your decision with others
  - How network structure influences cascades (ch. 19)
    - A third kind of cascade from the detailed structure of the network (neighbors and bridges)
- Structural properties
  - Power laws, rich-get-richer phenomena, and the long tail (ch. 18)
  - The small-world phenomenon (ch. 20)

Information cascades

Chapter 16

Following the crowd

- People connected in a network will influence each other
  - Mutual influence can sometimes have major effects
- Example of an information cascade: choosing restaurant A or B
  - Say you have done your own research and chosen restaurant A for dining
  - Arriving there, you see restaurant A is empty and restaurant B is full
  - It may be rational to join the crowd at B rather than follow your research
  - If each diner has independently decided that B is better, then collectively this may be stronger than your own research
  - It would then be reasonable for you to choose B contrary to your research
  - In this case, *herding* or an *information cascade* has occurred

Definition of an information cascade

- An information cascade has the potential to occur when people make decisions sequentially, with later people watching the actions of earlier people and inferring what the earlier people know
  - A cascade occurs when people abandon their own information in favor of inferences based on other people's actions
  - It is not mindless imitation! It is making rational inferences from limited information.
  - It is different from social pressure to conform, because of the inference
- There are two kinds of cascades
  - Information cascade versus direct-benefit cascade
  - In the former, it is inference that makes the cascade (today's lecture!)
  - In the latter, it is benefit to you that makes the cascade (we see it later in the course)
    - An example of a direct-benefit cascade is fax machines. A fax machine is only useful if other people possess a fax machine. If all your collaborators have fax machines, it is to your benefit to own a fax machine. No information about fax machines is needed.

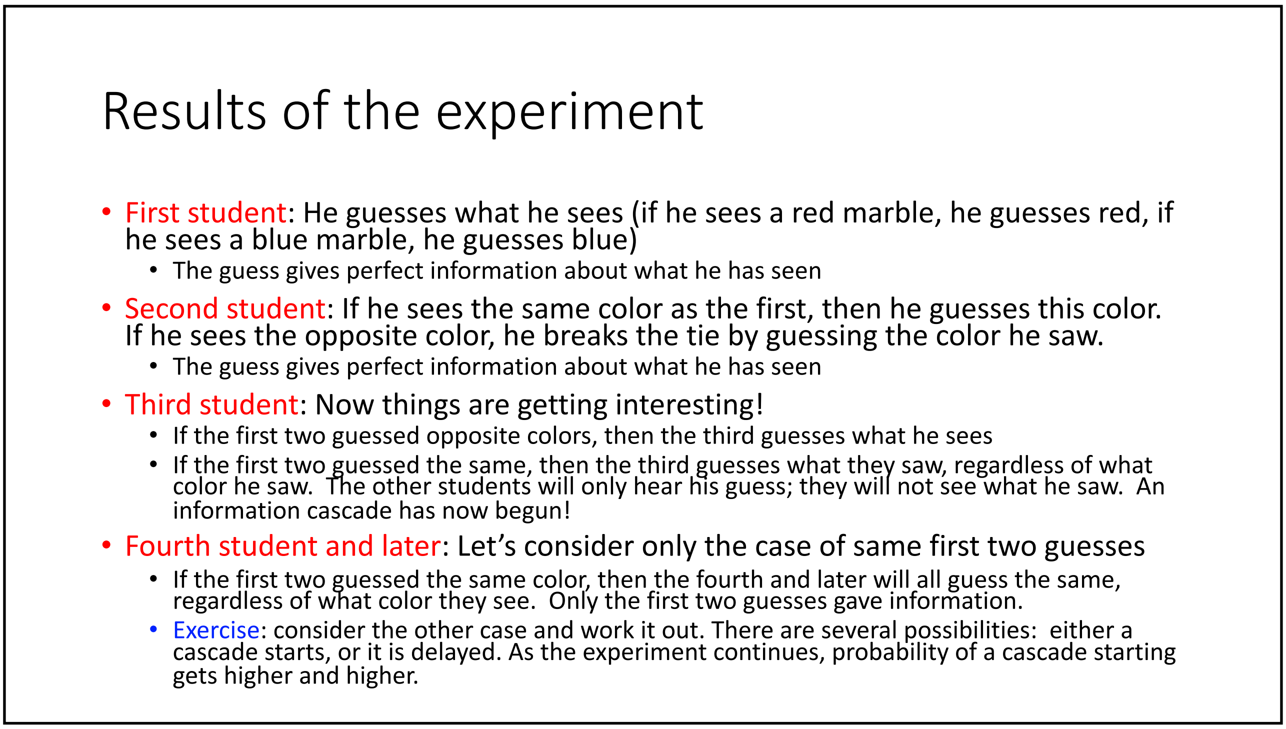
How to understand information cascades

- As we did for many other concepts in the course, we define an idealized model
  - The model can be completely understood
  - Consequences of the model will often hold also in more realistic cases
- A simple herding experiment
  - We introduce the concepts by means of a simple herding experiment
- Bayes' Rule
  - We explain Bayes' Rule which is used to compute conditional probabilities as part of the model
- Idealized model of cascades
  - We define an idealized model of sequential decision making where each participant has some private information to help his decision and also sees the decisions of the earlier participants (but not their private information)
  - We show exactly how cascades can arise in this model
- Realistic situations
  - We explain how realistic situations can differ from this model, but the main conclusions still hold

A simple herding experiment

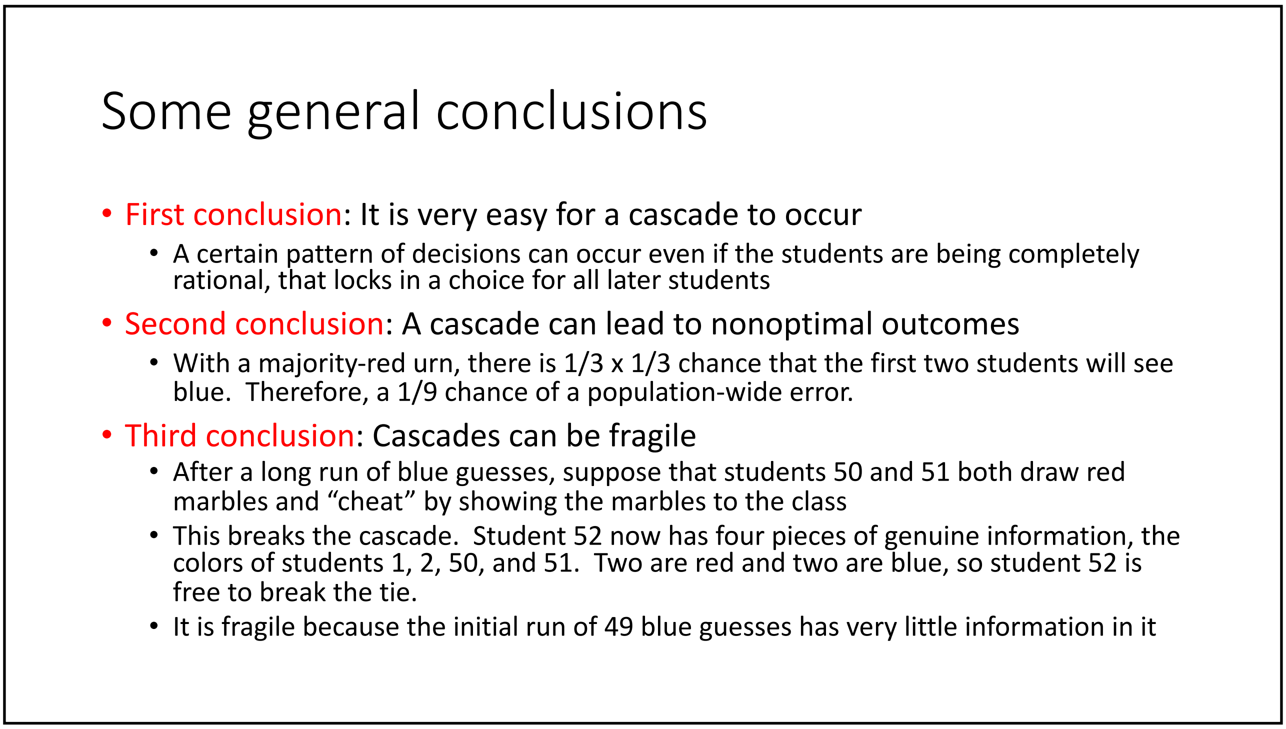
A simple herding experiment

- We start with a simple experiment to introduce the reasoning
  - This will help us understand the mathematical model that is introduced later
  - There is a decision to be made (adopt new technology, wear a new style, …)
  - People make the decision sequentially and each observes choices made earlier
  - Each person has some private information to help them decide
  - Each person makes inferences about others' private information from their actions
- Setup of the experiment:
  - A classroom with students
  - An urn at the front with three marbles
    - It contains either 2 red and 1 blue, or 2 blue and 1 red, with 50% chance of either
  - One by one, a student comes to the front, looks at one marble privately, then guesses publicly whether the urn is "majority red" or "majority blue"



Results of the experiment

- First student: He guesses what he sees (if he sees a red marble, he guesses red, if he sees a blue marble, he guesses blue)
  - The guess gives perfect information about what he has seen
- Second student: If he sees the same color as the first, then he guesses this color. If he sees the opposite color, he breaks the tie by guessing the color he saw.
  - The guess gives perfect information about what he has seen
- Third student: Now things are getting interesting!
  - If the first two guessed opposite colors, then the third guesses what he sees
  - If the first two guessed the same, then the third guesses what they saw, regardless of what color he saw. The other students will only hear his guess; they will not see what he saw. An information cascade has now begun!
- Fourth student and later: Let's consider only the case of same first two guesses
  - If the first two guessed the same color, then the fourth and later will all guess the same, regardless of what color they see. Only the first two guesses gave information.
  - Exercise: consider the other case and work it out. There are several possibilities: either a cascade starts, or it is delayed. As the experiment continues, probability of a cascade starting gets higher and higher.



Some general conclusions

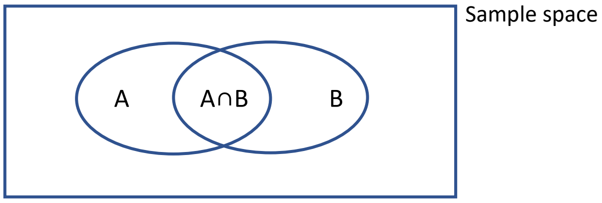
- First conclusion: It is very easy for a cascade to occur
  - A certain pattern of decisions can occur even if the students are being completely rational, that locks in a choice for all later students
- Second conclusion: A cascade can lead to nonoptimal outcomes
  - With a majority-red urn, there is 1/3 x 1/3 chance that the first two students will see blue. Therefore, a 1/9 chance of a population-wide error.
- Third conclusion: Cascades can be fragile
  - After a long run of blue guesses, suppose that students 50 and 51 both draw red marbles and "cheat" by showing the marbles to the class
  - This breaks the cascade. Student 52 now has four pieces of genuine information, the colors of students 1, 2, 50, and 51. Two are red and two are blue, so student 52 is free to break the tie.
  - It is fragile because the initial run of 49 blue guesses has very little information in it

Bayes' Rule

Conditional probability and Bayes' Rule

- To make a mathematical model of cascades, we have many questions of the form "What is the probability of X, given that I know Y and the previous person asserted Z?"
  - We need to make decisions when we have some partial information
  - To do this, we will use conditional probability and Bayes' Rule
- We compute the probabilities of events
  - An event is a set of possible outcomes of an experiment
  - A sample space is the set of all possible outcomes of an experiment
  - Every event A is a subset of the sample space. We assign a probability Pr[A] to each event, which can be interpreted as the fraction of outcomes in which A occurs when we repeat an experiment arbitrarily many times.
- Let us see how to compute conditional probabilities

Events in a sample space



- Given a sample space, we can depict events graphically
  - Event A is a region in this space: the set of all outcomes where A occurs
  - Given two events A and B, the joint event when both occur is shown by the area where they overlap. This is their intersection which is denoted A∩B.
- We need to consider the probability of A given that B has occurred
  - This is called the conditional probability of A given B and denoted by Pr[A|B]

Conditional probability and Bayes' Rule

- We compute the probability of A given B, and likewise B given A:

$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]} \qquad \Pr[B \mid A] = \frac{\Pr[B \cap A]}{\Pr[A]}$$

- From this we can compute Pr[A|B] in terms of Pr[B|A]:

$$\underbrace{\Pr[A \mid B]}_{\text{Posterior probability of A}} = \underbrace{\Pr[A]}_{\text{Prior probability of A}} \times \underbrace{\frac{\Pr[B \mid A]}{\Pr[B]}}_{\text{Effect of knowing B (correction)}}$$

Bayes' Rule

Example of Bayes' Rule

- We apply Bayes' Rule to assess the probability of a choice, given that we have some private information or observed some other events
- The example is about eyewitness testimony
  - In a particular city, 80% of taxi cabs are black and 20% are yellow
  - A witness to a hit-and-run accident says the taxi involved was yellow
  - Assume witnesses sometimes misidentify colors of cabs: if it is yellow they will say yellow 80% of the time, if black they will say black 80% of the time
  - What is the probability that the cab is yellow, if the witness says yellow?
- We are looking for Pr[true=Y | report=Y]
  - Probability that true color is Y if reported color is Y

Computing the probability of a yellow cab

- Applying Bayes' Rule, we get:

$$\Pr[\text{true=Y}|\text{report=Y}] = \Pr[\text{true=Y}] \times \frac{\Pr[\text{report=Y}|\text{true=Y}]}{\Pr[\text{report=Y}]}$$

- We know Pr[report=Y|true=Y]=0.8 and Pr[true=Y]=0.2
- We can figure out Pr[report=Y] as follows:

$$\begin{aligned}\Pr[\text{report=Y}] = {} & \Pr[\text{true=Y}]\times\Pr[\text{report=Y}|\text{true=Y}] + \\ & \Pr[\text{true=B}]\times\Pr[\text{report=Y}|\text{true=B}]\end{aligned}$$

$$\Pr[\text{report=Y}] = 0.2\times0.8 + 0.8\times0.2 = 0.32$$

- Therefore Pr[true=Y|report=Y] = 0.2 × (0.8 / 0.32) = 0.5

If witness says yellow, the cab is yellow with 50% probability! This substantially increases our belief that the cab is yellow, since with no information it would be 20%.

Example in spam filtering

- Bayes’ Rule is often used in e-mail spam detection
- Suppose you receive a piece of mail with subject “check this out”
  - What is the chance that the message is spam?
- This is a question about conditional probability:
  - Pr[message is spam | subject contains “check this out”]
  - Let’s just write Pr[spam|”check this out”]
- To compute this, we need some general information
  - Pr[spam]=0.4 *(40% of your e-mail is spam)*
  - Pr[“check this out”|spam]=0.01 *(1% of all spam contains “check this out”)*
  - Pr[“check this out”|nonspam]=0.004 *(0.4% of nonspam contains “check this out”)*

Computing the probability it is spam

- We can now proceed exactly like in the previous example:

$$\Pr[\text{spam}|\text{"check this out"}] = \Pr[\text{spam}] \times \frac{\Pr[\text{"check this out"}|\text{spam}]}{\Pr[\text{"check this out"}]}$$

- We need to compute the denominator, as before:

$$\Pr[\text{"check this out"}] = \Pr[\text{spam}]\times\Pr[\text{"check this out"}|\text{spam}] + \Pr[\text{nonspam}]\times\Pr[\text{"check this out"}|\text{nonspam}]$$

$$\Pr[\text{"check this out"}] = 0.4\times0.01 + 0.6\times0.004 = 0.0064$$

- This results in:

$$\Pr[\text{spam}|\text{"check this out"}] = 0.004/0.0064 = 0.625$$

- The presence of “check this out” is a signal that increases the likelihood that the message is spam. Realistic spam filters combine many signals, and each signal provides evidence. If many signals say it is somewhat likely to be spam, then it is almost surely spam.

Bayes' Rule in the herding experiment

How to use Bayes' Rule for herding

- We use Bayes' Rule to justify the reasoning used in the herding experiment
- Each student's decision is based on a conditional probability: whether the urn is majority-red or majority-blue, depending on the student's information
- The student should guess majority-blue if: (and majority-red otherwise)

$$\Pr[\text{majority-blue} \mid \text{student's information}] > 1/2$$

- We know the following probabilities:
  - $\Pr[\text{majority-blue}] = \Pr[\text{majority-red}] = 1/2$
  - $\Pr[\text{blue} \mid \text{majority-blue}] = \Pr[\text{red} \mid \text{majority-red}] = 2/3$
- With this knowledge we can calculate how the student should guess

How the students should guess

- We determine how the first student should guess:

$$\text{Pr[majority-blue|blue]} = \text{Pr[majority-blue]} \times \frac{\text{Pr[blue|majority-blue]}}{\text{Pr[blue]}}$$

Pr[blue] = 1/2 *(calculate it!)*
Pr[majority-blue|blue] = 2/3 *(calculate it!)*

- This is completely reasonable!
- We can also determine how the second student should guess:

Pr[majority-blue|blue,blue] = (…)

- We leave this as an exercise for you!
  - Hint: Pr[blue,blue|majority-blue]= Pr[blue|majority-blue]×Pr[blue|majority-blue] (because the two blue events are independent)

The third student: where the cascade forms

- We assume the first two students have annonced guesses of blue and the third student draws a red marble
  - The first two guesses convey genuine information, so the third student knows that there are three guesses blue, blue, red
- How the third student should guess:

Pr[majority-blue|blue,blue,red] = 2/3 *(using Bayes' Rule)*

- The third student ignores the red marble and guesses blue!
  - All future students will do the same: the cascade has formed!

A simple cascade model

A simple cascade model

- Let us now formulate a model for cascade formation
- Consider a group of people (numbered 1, 2, 3, …) who will sequentially make decisions to accept or reject some option
- The model contains the following elements:
    - Initially the world is in one of two states, G where the option is a good idea and B where the option is a bad idea, with Pr[G]=p and Pr[B]=1–p
    - Individuals receive payoffs depending on their decision. Rejecting the option gives 0, accepting gives $v_g>0$ if it is a good idea and $v_b<0$ if it is a bad idea.
    - Expected payoff with no extra information is equal to 0, that is, $v_g p + v_b(1–p) = 0$
    - Each individual gets a private signal before making his decision, H says that accepting is a good idea and L says that accepting is a bad idea, with Pr[H|G]=q>1/2 and Pr[L|G]=(1–q), Pr[L|B]=q>1/2 and Pr[H|B]=(1–q)
        - The signal tends to point to the right decision: H points to G and L points to B

Decision making with one signal

- Individuals decide based on two things: (1) their private signal, and (2) observation of the earlier decisions made by other individuals
- Let's first consider when the individual only has a private signal
- If a person gets signal H then their payoff shifts from:
  $v_g\Pr[G] + v_b\Pr[B] = v_g p + v_b(1-p) = 0$ *(before the signal)*
  to:
  $v_g\Pr[G|H] + v_b\Pr[B|H]$ *(after getting a signal H)*
- We use Bayes' Rule to determine the new expected payoff: *(calculate Pr[B|H]!)*
  $\Pr[G|H] = \Pr[G]\times(\Pr[H|G]/\Pr[H])$
  $\Pr[G|H] = \Pr[G]\times\Pr[H|G]/(\Pr[G]\times\Pr[H|G] + \Pr[B]\times\Pr[H|B])$
  $\Pr[G|H] = pq / (pq + (1-p)(1-q))$
- The new payoff is:
  $(v_g pq + v_b(1-p)(1-q)) / (pq+(1-p)(1-q))$
- Since $q>1/2$ and $(1-q)<1/2$, the new payoff is greater than 0 *(calculate it!)*
  - So the individual should accept the option when getting a high signal!

Decision making with multiple signals

- Let's now consider when the individual has more information
- Assume the individual gets a sequence S of independent signals containing a high signals and b low signals (interleaved of course)
- We can derive the following facts:
  i. $\Pr[G|S] > \Pr[G]$ when $a>b$ (posterior probability higher than prior probability)
  ii. $\Pr[G|S] < \Pr[G]$ when $a<b$ (posterior probability lower than prior probability)
  iii. $\Pr[G|S] = \Pr[G]$ when $a=b$
- Therefore the individual should accept the option when $a>b$, reject the option when $a<b$, and be indifferent when $a=b$

Justification of facts (i), (ii), and (iii)

- We compute Pr[G|S] using Bayes' Rule:

$$\Pr[G|S] = \Pr[G] \times \frac{\Pr[S|G]}{\Pr[S]}$$

- We note that $\Pr[S|G] = q^a(1-q)^b$ *(because signals are independent)*
- We calculate $\Pr[S] = \Pr[G]\times\Pr[S|G] + \Pr[B]\times\Pr[S|B]$
  - $\Pr[S] = pq^a(1-q)^b + (1-p)(1-q)^a q^b$
- We then get:

$$\Pr[G|S] = \frac{pq^a(1-q)^b}{pq^a(1-q)^b + (1-p)(1-q)^a q^b}$$

We need to compare this to p.
We know that q>1/2.
We see when a=b that it equals p.
What happens when a>b?

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Sequential decision making and cascades

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Sequential decision making

- Now we have made all the computations we need for the model
- Assume that individuals make decisions in sequence
  - Each individual sees only what earlier people *do*, not what they *know*!
  - Each individual has access to their own private signal plus the accept/reject decisions of all earlier people
  - They do not see the private signals of the earlier people!
- Now we can reason about the sequence of decisions
  - Sometimes, this can lead to a cascade, and sometimes it will not
  - We will see exactly when the cascade forms!

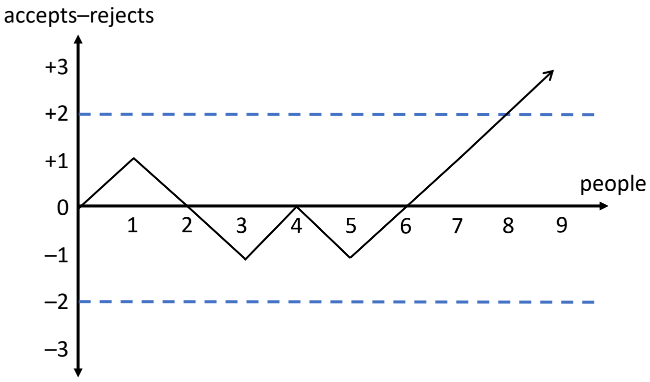
Decisions according to the model (1)

- Person 1 will follow his private signal
- Person 2 knows that person 1's decision reveals their private signal
  - It is as if person 2 gets two signals. If they are the same, follow them! If they are different, person 2 follows his private signal (tie break). So person 2 always follows his own signal.
- Person 3 knows that person 1 and person 2 both revealed their signals
  - It is as if person 3 has received three independent signals. Person 3 will follow the majority signal (H or L), according to calculations done earlier.
  - If person 1 and person 2 made opposite decisions, then person 3 will use his own signal as tiebreaker. Future people will know that this is useful information.
  - If person 1 and person 2 made the same decision, then person 3 will follow this. Future people will know that person 3's decision has no useful information. In this case, a cascade has begun!

Decisions according to the model (2)

- Let us now consider future people beyond person 3
- Person N (with N>3):
  - Suppose all earlier people followed their signals and person N knows this
  - If (number of accepts) = (number of rejects), then N will be tiebreaker and therefore follow his own signal
  - If (number of accepts) differs by one from (number of rejects), then N's signal will either make him indifferent or will reinforce the majority. Either way, N will follow his own signal.
  - If (number of accepts) differs by two or more from (number of rejects), then N's private signal will never change anything. It will never outweigh the previous majority. Furthermore, in this case, people N+1, N+2, etc., will know that N ignored his signal and will be in exactly the same position as N. Therefore, a cascade has begun.

How a cascade begins



- A cascade begins when the difference between the number of accepts and rejects reaches two
- As long as the number of accepts differs by the number of rejects by at most one, each person simply follows their private signal
- It is highly improbable that the difference remains forever in the narrow interval (from –1 to +1)
  - As the number of people increases without bounds, the probability of a cascade converges to 1

Applying the model to the real world

- This is an idealized model of sequential decision making
- In more realistic cases, people might see only some of the earlier decisions, or private signals might vary from one person to another, or payoffs may vary from one person to another
  - These more general cases are much more complicated to analyze, but the overall conclusions tend to be qualitatively similar
  - When people see others' ***public decisions*** but not their ***private knowledge***, then the population can tip into a situation when people, still behaving rationally, start following the crowd instead of their own information

Major enabler for cascades

Lessons from cascades

Lessons from cascades

- The observations we made about the herding experiment are reinforced:
  - Cascades can be wrong. If accepting the option is a bad idea, but the first two people get high signals, then a cascade of acceptances will start immediately, even though it is the wrong choice.
  - Cascades can be based on very little information. People will ignore their private information when a cascade starts: only the pre-cascade information influences the population's behavior. If the cascade starts quickly, then most of the private information is ignored!
  - Cascades are fragile. People who receive superior information can overturn even long-lived cascades (e.g., if they receive two private signals). Private signals that are made public can overturn cascades.
- Main lesson: be careful in drawing conclusions from the behavior of a crowd! A crowd can be wrong even if everybody is rational.
  - Second main lesson: if you are part of a cascade happening and you notice the cascade, then if you think it is wrong, then you can break it!

"The Wisdom of Crowds"?

- The book "The Wisdom of Crowds" by James Surowiecki seems to say the opposite: the aggregate behavior of many people with limited information can sometimes produce very accurate results
  - Take the average of people's guesses of the number of jelly beans in a jar
- The key to Surowiecki's argument is that the people are guessing independently, without knowing what the others have guessed
- If instead, they guess sequentially, and can observe the earlier guesses, then we are back to the cascade model and the result may be no good at all
  - Surowiecki notes this possibility as well!

“Tour de Table” (Around the Table)?

- In group decision making, one style (common in Belgium!) is to go around the table, asking people in sequence to express their opinion on option A or option B
- A cascade may quickly develop: if a few people initially favor A, others may be led to conclude that they should favor A, even if they initially favored B on their own
  - It is not just a case of social pressure! It may actually be rational decision making by the participants, as we saw in this lecture.
  - It can often be better to get the opinions independently in a first phase, and then try to reach consensus in a second phase

Summary

Summary

- We study decision making in groups of people
  - We study sequential decision making, where each person (1) has some private information, and (2) sees the decisions made by earlier people (but not their private information)
  - Using Bayes' Rule on conditional probabilities, we define a formal model for sequential decision making
- We show that even if people decide rationally, this can go wrong
  - It leads to information cascades, where a small amount of information will force everybody's decision to be the same (and it is often incorrect)
  - Another approach, independent decision making, does not lead to information cascades, so introducing independence is one way to avoid cascades
- We apply this to realistic situations
  - Cascades often occur even when situations differ from the formal model
  - However, cascades are fragile and it is usually possible to break the cascade if additional information is introduced into the decision making
  - In later lectures we will refine our understanding of cascades by adding direct benefits and by studying the effect of network structure (neighbors)

LINFO1115
Reasoning about
a highly connected world

Lecture 12
Direct-benefit cascades (network effects)

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Network dynamics

- Network dynamics focuses on how networks influence group behavior
- Cascades
  - Information cascades (ch. 16) Last week
    - Sequential decision making versus independent decision making
    - Bayes' Rule and an idealized cascade model explain how cascades form
  - Direct-benefit cascades (also called network effects) (ch. 17) ← *This lecture*
    - Another kind of cascade exists when aligning your decision gets you a benefit
    - This is very different from information cascades! We define an economic model to analyze it.
    - This leads to stable and unstable equilibria and tipping points
  - How network structure influences cascades (ch. 19)
    - Cascading is influenced by the detailed structure of the network (neighbors and bridges)
- Structural properties
  - Power laws, rich-get-richer phenomena, and the long tail (ch. 18) ← *This lecture*
  - The small-world phenomenon (ch. 20)

Direct-Benefit Cascades (Network Effects)

Chapter 17

Direct-benefit cascades (network effects)

- There are two reasons why individuals might imitate others
  - Informational effects: observing the behavior of other people gives information about what they know, so you can use the observations to influence your decision
  - Direct-benefit effects (also called network effects): sometimes there is an explicit benefit to aligning your behavior with that of others
- Network effects arise often in adoption of new technologies
  - Introduction of the fax machine: its value depends on others using it
  - Social networking or media-sharing application: value increases with more users
  - Computer operating system (Windows, Mac OS): even though not made to interact with others, its value still increases when there are many users (more software, etc.)
  - Word processing (Microsoft Office): exchange of documents is easy when everybody uses the same software

Network effects as “externalities”

- We will study network effects by introducing concepts from economics
- In economics, the effects we describe are called positive externalities
    - An externality is a situation where the welfare of an individual depends on the actions of other individuals, without a mutually agreed-upon compensation
- There exist also negative externalities
    - For example traffic congestion, which we saw before, because your use of the network decreases the payoff to other users, and there is no compensation to them
- To be an externality, the effect has to be uncompensated
    - If you drink a can of Coke, there is one less can for others to consume, but since you pay for the can, which also pays for the fabrication of another can, you have compensated the other users. There is no uncompensated effect, so no externality.

Studying network effects

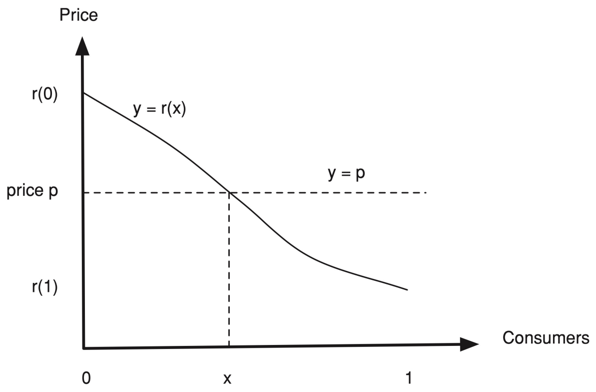
- We study network effects by defining an idealized economic model
    - We analyze the model and then apply the intuitions to the real world (our usual approach!)
    - Earlier, we explained matching markets with **discrete models**, but today’s are **continuous models** ⚠
- We will introduce four models, progressively adding concepts
    - First model: Economy without network effects, where users do not care how many other users there are. This is the base case for the market economy.
    - Second model: Economy with network effects, where more users increases the value to users and users know the fraction of the population using the product. This is a static model that leads to a self-fulfilling expectations equilibrium.
    - Third model: Economy with network effects, where users have an incorrect common belief of the fraction using the product. The incorrect belief introduces a dynamic behavior over time, as users correct their error. It leads to instability and stability: some equilibria are stable and some are unstable.
    - Fourth model: Economy with both individual and network effects, where the product is useful even to the very first user. This leads to a new stable equilibrium for a very low number of users, and a possibility to jump from one equilibrium to a higher equilibrium.

First Model:
The Economy without Network Effects

A market without network effects

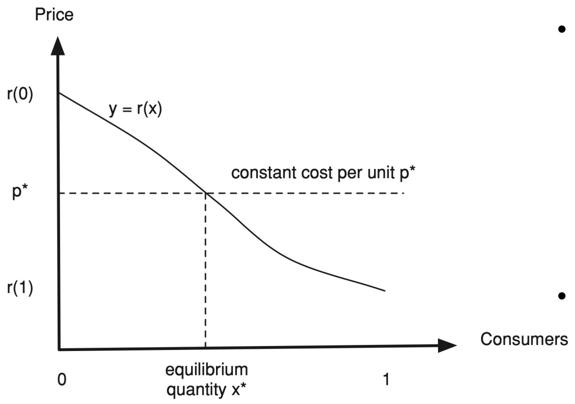
- We first look at the market when there is no network effect
- All consumers are given names as real numbers from 0 to 1
    - Consumers are uniformly distributed from 0 to 1, so the total mass of consumers is 1, and the interval [0,x] contains a fraction x of consumers
    - This is a continuous approximation to a market with a very large, but finite number of consumers. We assume that each consumer's decision whether to buy or not has negligeable effect on the price.
- Each consumer has a reservation price, which is the maximum amount he is willing to pay for one unit of the product
    - Let r(x) denote the reservation price of consumer x
    - Reservation prices are arranged in decreasing order: if r(x')>r(x), then x'<x

Relationship between price and demand



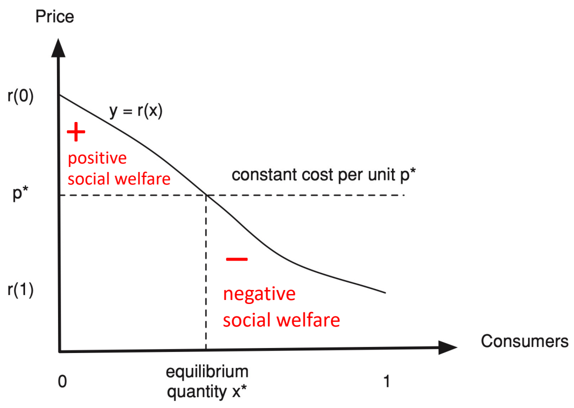
- Suppose the market price for a unit of the product is p
  - Everyone whose reservation price ≥ p will buy
  - Everyone whose reservation price < p will not buy
  - If the price were r(0) or more, nobody would buy
  - If the price were r(1) or less, everyone would buy
- Let us assume that r(1) < p < r(0)
  - There is a unique number x such that r(x)=p
  - All consumers from 0 to x buy the product
    - The fraction of consumers that buy is x
- The function giving x in terms of p (inverse of r) is called the demand function in microeconomics
  - How many consumers will buy as function of the price

Equilibrium quantity of the product



- Suppose the production cost is p* per unit and there are many potential producers so none is big enough to influence the price
  - Producers can produce any amount at price p* and none below p*
  - Price cannot remain above p* because of competition!
    - Market price is p*
    - Equilibrium quantity of the product is x*
      - The unique x* such that r(x*)=p*
- This is really an equilibrium!
  - If less than x* fraction purchased, there would be consumers with incentive to buy: “upward pressure” on consumption
  - If more than x* fraction purchased, there would be consumers who bought but did not want to: “downward pressure” on consumption

Social optimality of the equilibrium



- It is interesting to see that this equilibrium is socially optimum
  - Remember the definition (in the lecture on game theory): a choice of strategies (here, buy or not buy for each consumer) is socially optimal if it maximizes the sum of the consumers' payoffs
- Computing the social welfare
  - Payoff for a consumer x' is r(x')-p*
  - Social welfare for all consumers from 0 to x is the area (positive and negative!) between the curve y=r(x) and the line y=p*
  - We collect only the positive area, which means to collect from consumers 0 to x* (the others don't buy)
- Note that it is the integral $\int_0^{x^*}(r(x)-p^*)dx$

Second Model: The Economy with Network Effects

A market with network effects

- With network effects, a potential consumer takes into account their own reservation price and also the total number of users
  - For simplicity, we assume the customers correctly know how many users there are
- There are now two functions at work:
  - r(x): the intrinsic reservation price for consumer x (same as before)
  - new → f(z): the benefit to each consumer when fraction z has bought the product
- The actual reservation price of consumer x is r(x)f(z), he buys if r(x)f(z)≥p*
  - f(z) is increasing in z: benefit increases when more people use the product
  - It’s a multiplication because people with higher intrinsic value r(x) benefit more
  - We also assume f(0)=0, if nobody purchases then benefit is zero
  - Finally, we assume that f(z) is a continuous function (approximation!)
  - For simplicity, we assume that r(1)=0, some consumers will only pay very little

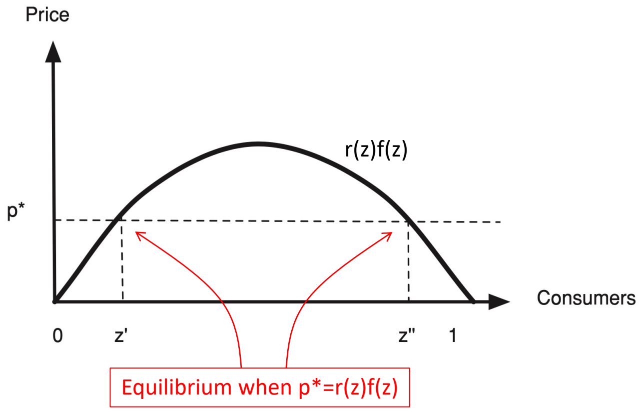
Equilibria with network effects

- We assume that all consumers expect that the fraction of the population buying the product is z, and they know what is the fraction actually buying
  - In that case, the fraction who actually purchase the product will be z!
  - This is called a self-fulfilling expectations equilibrium
- If everybody expects z=0 to buy, then r(x)f(0)=0 and nobody will buy!
- What happens if 0<z<1?
  - If fraction z buys the good, what consumers does this correspond to?
  - If consumer x’ buys and if x<x’, then x will buy too (since r(x)f(z) is higher)
  - Set of purchasers is exactly the consumers from 0 to z, and the price is the price of consumer z (the lowest) who has a reservation price of r(z)f(z)
  - So the equilibrium price is p*=r(z)f(z)

Comparing the two models

- We compare the model without network effects and the model with network effects
  - Equilibrium price is p*=r(x*) in the model without network effects
  - Equilibrium price is p*=r(z)f(z) in the model with network effects
- Without network effects, when we lower the price p*, the number of consumers buying increases, which is x*. Simple!
- With network effects, it's not so simple
  - One equilibrium exists for z=0
  - Do other equilibria exist? To answer this question, we need to know the shape of the functions r(.) and f(.)

Concrete example of network effects

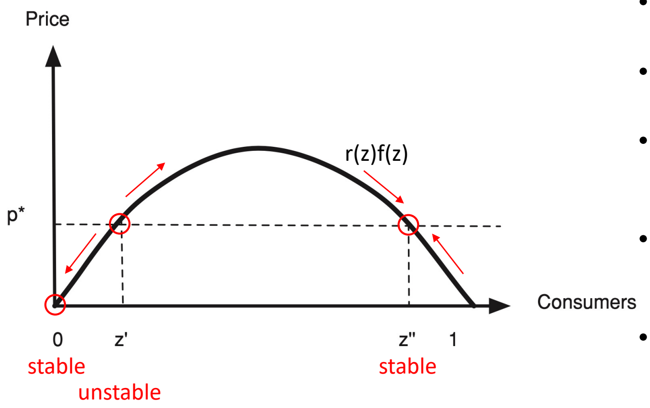


Equilibrium when p*=r(z)f(z)

- Let's say that r(x)=1–x and f(z)=z
  - Reasonable? Yes! r(.) decreases from 1 to 0 and f(.) increases with f(0)=0.
- So r(z)f(z) = z(1–z) which is parabolic
  - 0 at z=0 and z=1, max 1/4 at z=1/2
  - In general, r(.) and f(.) are not exactly like this, but with same general shape
- If p*>1/4 there are no equilibria
- If 0<p*<1/4 there are two equilibria
  - This gives three equilibria in all, 0, z', z"
- For each of these three values, if people expect fraction z to buy, then precisely fraction z will buy!

Stability, instability, and tipping points

Exploring the equilibria



- Let’s continue to explore this example to understand what happens over time
- Suppose fraction z purchases, where z is not one of the three equilibria
- If 0<z<z’ then there is “downward pressure” on consumption: r(z)f(z)<p* so purchaser z is unhappy (he paid too much)!
- If z’<z<z” then there is “upward pressure” on consumption: r(z)f(z)>p* so consumers slightly higher than z want to buy
- If z”<z then there is again “downward pressure”: r(z)f(z)<p* so purchaser z is unhappy

Unstable equilibrium z’ is a *critical point*, also called a *tipping point*: we want to move past it

Effect of changing the price p*

- If p* is lowered:
  - Critical point z’ moves to the left, closer to 0, so easier to get past
  - High equilibrium z” moves to the right, so user population z” increases
- If p* is set below the cost of production, the firm loses money!
  - But this can be part of a long-term pricing strategy
  - Early losses are needed to achieve the high equilibrium
  - Later profits can offset the early losses
- Many firms do this
  - Free trials for their products
  - Low introductory prices

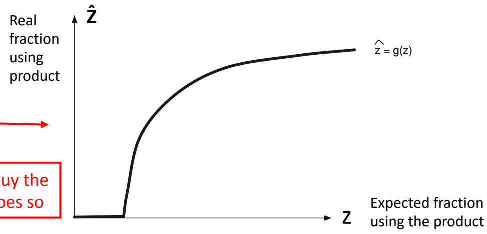
Third Model:
A Dynamic View
of the Market

A dynamic view of critical points

- There is another way to view critical points that is especially nice
  - So far, we assumed that consumers correctly predict the number of users
  - Let's see what happens when the customers have a common belief about the number of users, but this belief might be incorrect
- Assume everybody believes that fraction z will use the product
  - Consumer x will want to purchase if $r(x)f(z) \geq p^*$
  - All the consumers who want to purchase will be between 0 and $\hat{z}$, where $\hat{z}$ solves the equation $r(\hat{z})f(z)=p^*$
    - Consumers **believe fraction z** use the product, but the **real fraction is $\hat{z}$**
- Therefore $\hat{z} = r^{-1}(p^*/f(z))$
  - We can compute $\hat{z}$ from the shared belief z, but only if there is a solution for $\hat{z}$
  - Otherwise the outcome is simply that nobody purchases, i.e., $\hat{z}=0$

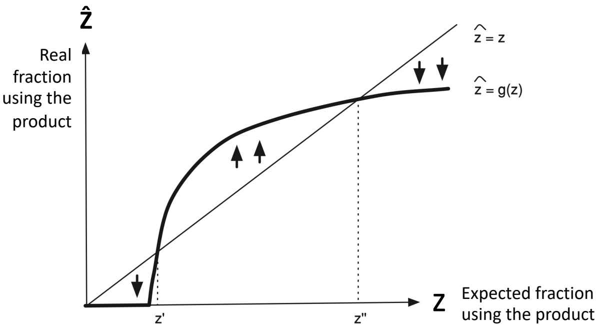
Computing $\hat{z}=r^{-1}(p^*/f(z))$

- Since r(.) is a continuous function that decreases from r(0) to r(1)=0
  - A solution $\hat{z}$ exists and be unique when $p^*/f(z) \leq r(0)$
- We define the function g(.) that gives $\hat{z}$ in terms of z as follows:
  - $g(z) = r^{-1}(p^*/f(z))$ when $p^*/f(z) \leq r(0)$, otherwise $g(z)=0$
- Let's compute g(z) for our example where $r(x)=1-x$ and $f(z)=z$
  - We know $r^{-1}(x)=1-x$ and $r(0)=1$, so the condition becomes $z \geq p^*$ (calculate it!)
- Therefore:
  - $g(z) = 1 - p^*/z$ when $z \geq p^*$, otherwise $g(z)=0$
- Let's plot this curve



If everybody expects fraction z to buy the product, then fraction $\hat{z}$ actually does so

Outcome ẑ versus expectations z

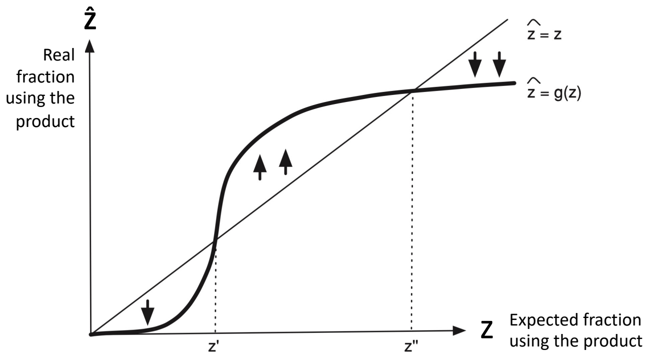


unstable

stable:
upward from below and
downward from above

- It's interesting to see the relationship of this curve to the 45° line ẑ=z
  - When ẑ=g(z) and ẑ=z intersect, we have a self-fulfilling expectations equilibrium
  - When ẑ=g(z) lies below the line ẑ=z, we have downward pressure on the consumption: the outcome underperforms!
  - When ẑ=g(z) lies above the line ẑ=z, the outcome overperforms so upward pressure!
- This intuition continues to hold in many realistic cases…

Same behavior in more general cases



unstable

stable:
upward from below and
downward from above

- The same behavior happens in more general cases
- The curve qualitatively looks the same, except it is smoother

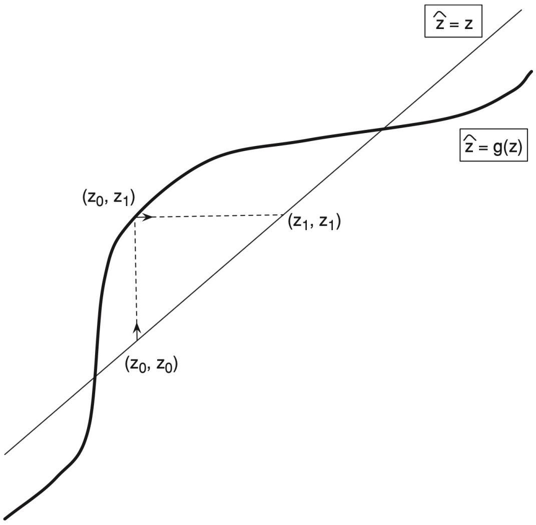
Dynamic behavior of the population

- We can use these curves to show behavior over time
- Here's a scenario: (formulated in terms of social media)
  - People evaluate their participation in a large social media site
  - Each person x has an intrinsic interest in using the site, with function r(x)
  - The site's attractiveness increases with more users, with function f(z)
  - There is a fixed level of effort required to use the site, which plays the role of a "price" p* (expending effort rather than money)
  - If person x expects fraction z to participate, then x participates if $r(x)f(z) \geq p^*$
- Time advances in periods t = 0, 1, 2, etc.
  - At time t=0, the initial audience size is z_0

Audience size changing over time

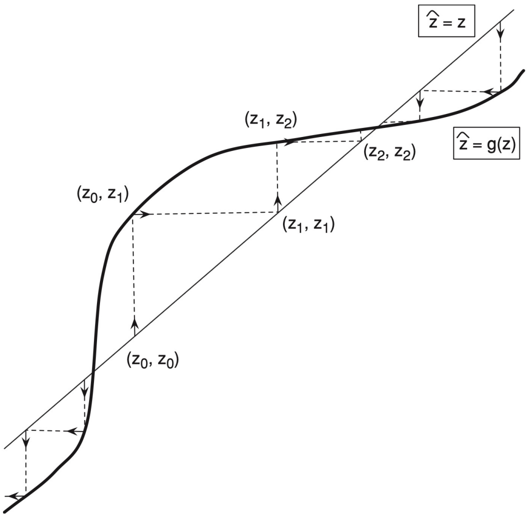
- Time advances in periods t = 0, 1, 2, etc. (days or weeks)
  - At time t=0, the initial audience size is z_0
- The audience size changes dynamically over time
  - In each period t, people evaluate whether to participate by expecting the audience size will be the same as it was in the previous period
  - Therefore $z_1 = g(z_0)$ since everyone acts in period t=1 by expecting z_0
  - In the next period, $z_2 = g(z_1)$ since everybody expects z_1, and so forth…
  - In general, we have $z_t = g(z_{t-1})$ for all t
- This is an approximation!
  - The people are being myopic: they expect the future to be like the past
  - But it is a reasonable approximation

Graphical analysis of dynamics



- We will analyze pictorially what happens with z_0, $z_1=g(z_0)$, $z_2=g(z_1)$, and so forth
- We track the points (z_t, z_t) on the line $\hat{z}=z$
- They follow a zigzag path
  - Up from $\hat{z}=z$ to $\hat{z}=g(z)$, then across to $\hat{z}=z$ again
  - This moves from (z_0, z_0) to (z_1, z_1), and so forth
- We can now graphically follow the time behavior of z!
  - This is an intuitive way to understand what happens but it is still completely rigorous

Graphical analysis of stability and instability



- We can determine stability and instability by tracking the zigzag path
  - It depends on the starting point
- We can see that the lower intersection is unstable
  - Zigzag paths move away from it
- We can see that the higher intersection is stable
  - Zigzag paths move toward it

Industries with network goods

Marketing a product with network effects

- How can a firm that wants to sell a product with network effects use these insights to market its product?
- Suppose you sell new software, hardware technology, or social media
  - The marketing will not succeed unless you get past tipping point z’
  - Starting small and hoping to grow is very unlikely to succeed
- You need to convince a large initial group to adopt your product
  - You could set a low introductory price for the product, perhaps for free
  - This results in early losses, but if you get past the tipping point, then you can raise prices and possibly make enough profit to overcome the initial losses
- An alternative is to identify “fashion leaders” (influencers)
  - Users whose purchase will influence other to use it. This requires identifying a network of connections: we will explore this idea in Chapter 19!

Social optimality with network effects

- We saw that for a market with no network effects, the equilibrium is socially optimal
- With network effects this is no longer the case
  - It is because adding a consumer also affects all the other consumers!
  - Suppose we are at an equilibrium z*
  - Consumer with name z* has a reservation price of r(z*)f(z*) = p*
  - Consider the consumers with names in (z*,z*+c) for a small constant c
  - None of these consumers want to buy
  - But if they did buy, then all current purchasers would benefit: the value for all consumers x < z* would increase from r(x)f(z*) to r(x)f(z*+c)
- The general principle is that for goods with network effects, markets usually provide less of the good than is socially optimal

Network effects and competition

- What happens when multiple firms develop competing new products, each of which has its own network effects?
  - The rise of Microsoft to dominate the personal computer industry
  - The triumph of VHS over Betamax as standard videotape format in the 1980s
- In the case of competition with network effects, it is likely that one product will dominate, as opposed to a scenario where both flourish
  - The product that first gets over its tipping point is usually the winner
  - Being first is much more important than being the best!
- If product A dominates over product B, what can B do?
  - Sometimes, if product B is massively improved and product A does not change, then B may still overtake A. But it is a long shot.

Fourth Model:
Mixing Individual Effects with Population-Level Effects

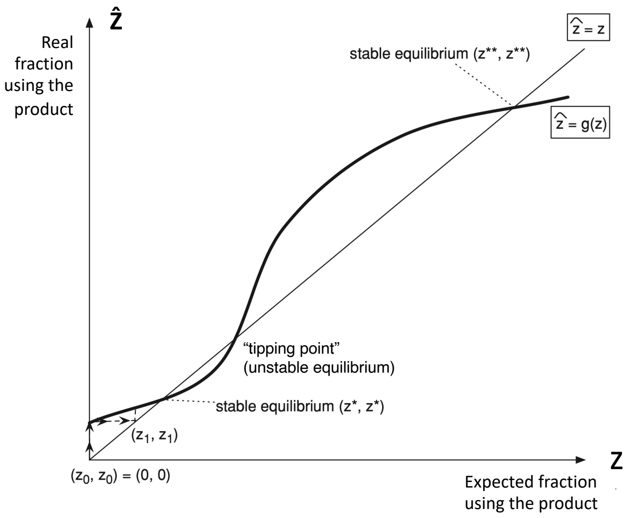
Products with value for the first user

- In the previous models, we assumed the product is useless when there are zero users (that is, we assumed f(0)=0)
  - Let us now assume the product has value even for the first user
  - We assume f(0)>0 and we keep that f(z) is increasing in z
- We will see that this can add new qualitative phenomena!
  - It can result in a new stable equilibrium for low values of z
  - It can make it possible for a firm to grow its audience from 0 upwards
  - It can result in quick “flips” with huge increases of z over short time periods

A concrete model

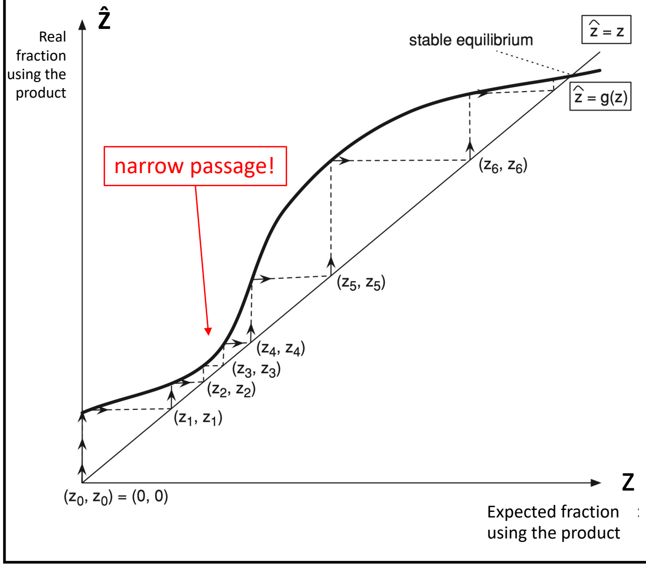
- Let’s consider the function $f(z)=1+az^2$ for constant parameter a
  - We keep the simple function $r(x)=1–x$
- We apply same analysis we did before to get the dynamic behavior
  - When everybody expects audience z, the actual audience is $\hat{z}=g(z)$
  - Calculate g(z) as before! *(exercise for you!)*
- This gives:
  - $g(z) = 1 – p^*/(1+az^2)$
- Let us plot this function together with the 45^o line $\hat{z}=z$ …

A small stable equilibrium appears



- In this new situation, we can grow from $z_0=0$ to (z^*,z^*) which is stable
  - The audience can grow from zero up to some larger stable number
- But it gets even better…
  - It’s possible to shift to the much higher stable equilibrium (z^{**},z^{**})

Bottlenecks and large shifts



- The firm would like to shift from z* to z** (of course!)
  - But there is an obstacle, a nasty “bottleneck”, that stops it
- There is way around the obstacle!
  - If the firm lowers the price slightly, to q*<p*, then we get a new function h(z) that is higher
  - At some point, the curve ẑ=h(z) no longer intersects the line ẑ=z
  - At this point, the zigzag line starts traveling upwards until it reaches z**
    - It slows down when it goes through the narrow “passageway” (the passageway used to be a real bottleneck when it was an unstable equilibrium)

Summary of direct-benefit cascades

Summary

- We introduced four models in steps to understand network effects
  - Economy without network effects (base case: “supply equals demand”)
    - Purchasers do not take into account whether other users purchase or not
  - Economy with network effects (static view, self-fulfilling equilibrium)
    - The value of an item increases when there are more purchasers
  - Economy with network effects (dynamic view, with both stable and unstable equilibria)
    - Purchasers may have an incorrect belief of the total number of purchasers
    - This gives dynamic behavior with stable equilibria and unstable equilibria (tipping points)
  - Economy with network effects (individual effects plus network effects)
    - The product has value even for the first purchaser
    - This adds a new stable equilibrium close to zero, and sometimes it can “jump” to a high one
- We studied how the intuitions of these models apply in the real world
  - How companies succeed when their product requires a big user base
  - Remember this when you make your own company!

Power Laws and Rich-Get-Richer Phenomena

Chapter 18

Network approach

- Cascades happen because people's decisions are connected
  - In the last two chapters we have seen how a person's decisions depend on the choices made by other people (information cascades and network effects)
  - Coupled decisions can lead to outcomes very different from individuals making independent decisions
  - We analyzed this by models that connect people together ("network approach")
- We now apply the network approach to the concept of popularity
  - Popularity has extreme imbalances: most people are unknown outside of their immediate circles, yet a few people achieve wider recognition, and a very few achieve global name recognition
  - Basic models of network behavior can give insight into these questions

Popularity of Web Pages

Popularity of a Web page

- We focus on the Web as a concrete domain in which we can measure and study popularity accurately
- Popularity of a Web page can be measured as the number of in-links
- This leads to a basic question:
  - What fraction of pages have k in-links, as a function of k?
- Studying this will lead to some unexpected results
  - Naïve models of the Web lead to incorrect results
  - A good model, preferential attachment, shows connections between pages

Power Laws

Naïve hypothesis for Web popularity

- Let's make a naïve model to see what we expect as an answer
- A natural guess is that Web page in-links are distributed according to a normal or Gaussian distribution, characterized by two values, a mean µ and a standard deviation σ
  - Probability of observing a value exceeding µ by more than c times σ decreases exponentially in c

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

- The normal distribution is ubiquitous across the natural sciences
  - The Central Limit Theorem states that if we take any set of small independent random quantities, then in the limit their sum (or average) will follow a normal distribution
  - This means the normal distribution is a very natural one

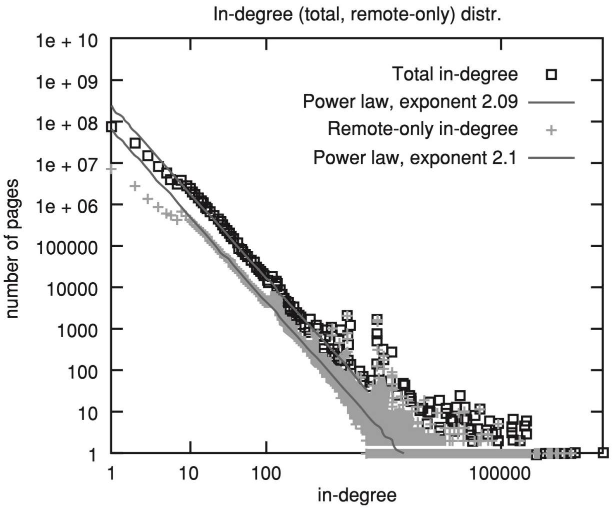
Empirical study of Web popularity

- Assume that each page decides independently and randomly whether to link to any other given page
  - The number of in-links is the sum of many independent random quantities
  - If this is correct, then the number of pages with k in-links should decrease exponentially in k, as k grows large
- However, this is completely contradicted by empirical studies
  - Studies of Web snapshots show that the fraction of Web pages with k in-links is approximately proportional to $1/k^2$ (or with exponent slightly higher than 2)
  - This is vastly different from the normal distribution: $1/k^2$ decreases much more slowly as e^{-k}
  - This form of empirical distribution is called a power law

Power laws

- Just as the normal distribution dominates in repeated measurements of quantities in the natural sciences, power laws such as $1/k^2$ dominate in settings where the quantity being measured is a kind of popularity
  - Fraction of telephone numbers that receive k calls per day ($1/k^2$)
  - Fraction of books that are bought by k people ($1/k^3$)
  - Fraction of scientific papers that receive k citations ($1/k^3$)
- Power laws can be tested easily
  - If you want to know whether a function f(k) has the form a/k^c, for some constants a and c, it suffices to make a log-log graph:
    log f(k) = log a – c log k
  - If this results in a straight line, then the function is a power law

Distribution of Web in-links



- This figure shows the in-degree versus the number of Web pages on a log-log plot
- It is approximately a straight line, and this only breaks down for a small number of pages with very high in-degrees
- How can we explain this? How can we explain that the line is so straight over so many orders of magnitude? What is the underlying process?

Preferential Attachment Model (Rich-Get-Richer)

Motivation for preferential attachment

- Just as normal distributions arise from many independent decisions averaging out, we will see that power laws arise from many correlated decisions across a population
  - Independent decisions ⇒ normal distribution
  - Correlated decisions ⇒ power laws

- We will define a simple model in which people have a tendency to copy decisions made by people before them
  - This model is called preferential attachment and it leads directly to a power law distribution

Preferential attachment model

- Here is a model for the creation of links among Web pages:
  1. Pages are created in order, named 1, 2, 3, …, N
  2. When page j is created, it makes a link to an earlier page by choosing randomly between actions (a) and (b):
     a) With probability p, page j choose a random page i uniformly from among all earlier pages, and creates a link to this page i
     b) With probability 1–p, page j instead chooses a random page i uniformly from among all earlier pages, and creates a link to the page that i points to
  - This defines the creation of a single link from page j; we can repeat this process to create multiple, independent links from page j
- Action (b) is the key: the author of page j will copy the decision made by the author of page i
- With this model, we can show that the fraction of pages with k in-links will be proportional to $1/k^c$ where the value of c depends on the probability p

Why is this called "rich-get-richer"?

- When you copy the decision of a random earlier page, the probability that you will link to some page m is directly proportional to the total number of pages that link to m
  - The more pages that link to m, the bigger chance you will hit on a random page that links to m!
- Action (b) could be rewritten as follows:
  - With probability 1–p, page j chooses a page m with probability proportional to m's current number of in-links, and creates a link to m
- Pages that are "rich", with many in-links, tend to get "richer", get more in-links!
  - It's intuitive: somebody who is popular will tend to get more popular

Growth over time amplifies small differences

- A page's popularity will grow at a rate proportional to its current value, i.e., exponentially with time
  - A page that gets a small lead over others will tend to extend its lead
- Compare this to independent random variables
  - Small independent random variables tend to cancel each other out, which is the crux of the Central Limit Theorem
  - Here, instead, differences are amplified
- This kind of growth is everywhere
  - Fraction of cities with population k is proportional to $1/k^c$
  - Number of copies of a gene in a genome follows a power law

Unpredictability of rich-get-richer effects

- For any popular item (Web page, book, song, etc.), the initial phase of its rise to popularity is fragile
  - It is sensitive to unpredictable initial fluctuations
  - Once it has become popular, it is no longer fragile but robust
- A worse technology can win because it reaches a certain critical size before its competitors do
  - If we replay the history of the Harry Potter books, would they again sell 100s of millions of copies, or would some other children's fiction achieve success?
- Experiments were done with music download sites with 48 obscure songs
  - Visitors were assigned to eight parallel sites, which started identically
  - It was found that the market share of songs varied considerably (although the best songs were never at the bottom and the worst were never at the top)

Relation to information cascades

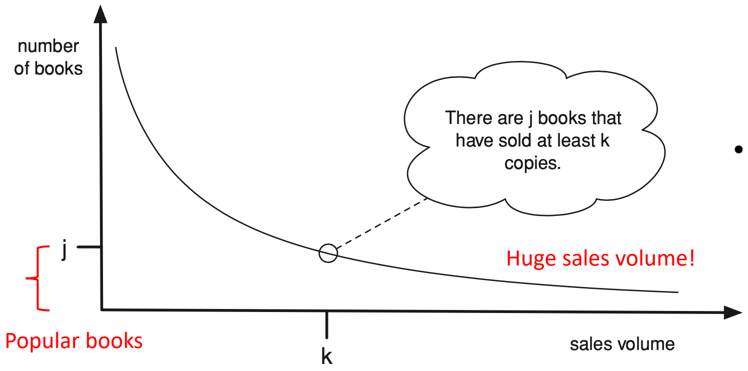
- Remember information cascades: people aware of earlier decisions could end up in a cascade
- Preferential attachment is similar but differs in important ways:
  - There are many possibilities to choose from (all Web pages) not just two
  - The new Web page looks at only one previous choice, not all previous choices
  - There is no discussion of "rationality", i.e., using rational reasoning
- This is still a research topic
  - How does individual rationality enter in the picture for Web popularity?
  - What about competing information cascades, whose intensities vary according to power laws?

The Long Tail

The long tail

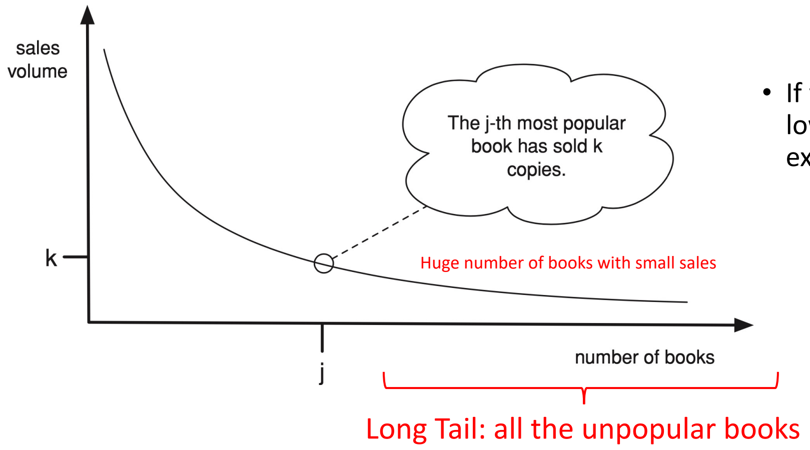
- The popularity distribution has major consequences in economics
  - For example, a media company with a large inventory (books or music): are most sales coming from a small set of enormously popular items, or from a much larger population of less popular items?
  - In the former case, the company is betting on "hits"
  - In the latter case, the company is betting on numerous "niche products"
- Internet-based distribution drives the media industry to the latter
  - Famous article from 2004, "The Long Tail," by Chris Anderson
  - A "long tail" of obscure products drives the bulk of audience interest
  - This is in accord with the business models of Amazon and Netflix: the ability to carry huge inventories is a big reason for their success. Huge inventories are much easier to implement on a Web site than in a brick-and-mortar store.
  - This is not completely new: supermarkets have known this for a long time, since they stock not only popular products but also many products with low popularity

Looking at popular books…



- How many items have sold at least k copies?
  - Very few items with large k
  - Many items with small k
- This graph focuses on the popular books, with huge sales volume
  - A very few books, such as the Harry Potter series, with enormous sales

Looking at unpopular books…



- We can switch our view of this graph to focus on the books with low sales
  - There are a huge number of them, together they are called the Long Tail
- If we concentrate on the books with low sales volume, we see that they extend very far out to the right
  - The sales volume goes down very slowly, because it is a power law!
  - The total sales of all unpopular books taken together, is often larger than the sales of the popular books

Summary of power laws and their effects

Summary

- Web page popularity follows a power law
  - This is because Web page creation is correlated and not independent
  - The preferential attachment model explains the power law
- This is a special case of a rich-get-richer phenomenon
  - Popularity grows exponentially, so small differences get bigger
  - Initially, it is very unpredictable, subject to random fluctuations
  - When popularity is large it is more robust
- The long tail
  - Power laws mean that the popularity decreases very slowly
  - So a huge number of unpopular pages can have large aggregate popularity
  - Success of many Web sites depends on having a huge inventory

LINFO1115
Reasoning about
a highly connected world

Lecture 13
How network structure influences cascades

Peter Van Roy

Academic year 2022-23
École Polytechnique de Louvain
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Final lecture and course summary

- Today's lecture is the final lecture of the course
- Recall that the course is organized in five parts:
  - Graph theory (chapters 1-5): connectivity and structural balance
  - Game theory (chapters 6, 8, 9): strategies, traffic, and auctions
  - Markets (chapters 10-12): markets and how networks influence them
  - World-Wide Web (chapters 13-15): structure of the Web and search engines
  - Network dynamics (chapters 16-19): how networks influence people's behavior
- We defined many idealized formal models of real-world activities
  - This gives understanding and intuition while keeping things simple
  - Defining simple models is a good approach to understand complex systems!

The fifth part: network dynamics

- Network dynamics focuses on how networks influence group behavior
- Cascades
  - Information cascades (ch. 16) Previous two weeks
    - Sequential decision making versus independent decision making
    - Bayes' Rule and an idealized cascade model explain how cascades form
  - Direct-benefit cascades (also called network effects) (ch. 17)
    - Another kind of cascade exists when aligning your decision gets you a benefit
    - This is very different from information cascades! We define an economic model to analyze it.
    - This leads to stable and unstable equilibria and tipping points
  - How network structure influences cascades (ch. 19) *Today's lecture*
    - Cascading is influenced by the detailed structure of the network (neighbors and bridges)
- Structural properties
  - Power laws, rich-get-richer phenomena, and the long tail (ch. 18) Previous week
  - The small-world phenomenon (ch. 20) *(not given this year)*

How Network Structure Influences Cascades

Chapter 19

Decision making in networks

- Previously we studied how a person's choices depend on what other people do
  - In the last two lectures we have looked at information cascades, direct-benefit cascades, and rich-get-richer dynamics to model how this works
  - In those lectures we looked at the network as a homogeneous set of people, without taking into account how these people are connected
- → Today we will look at how choices are affected by network connections
  - We look at the network as a graph with explicit connections
  - This gives a new kind of cascade that diffuses through the network
- This lets us understand some phenomena that can't be modeled when we look only at homogeneous populations
  - Many of our decisions depend on friends and colleagues. We choose technology they use, we often align our ideas with their ideas.
  - This goes back to Chapter 4 when we introduced homophily (similarity with friends). Homophily only tells part of the story; we also need to know the network graph.

Diffusion in Networks

Diffusion of innovations (1)

- In sociology, the diffusion of innovations has been long studied
  - How new technologies or ideas spread through a group of people
  - Usually, the innovation diffuses in two waves: in the first wave it is informational, and in the second wave it is direct-benefit
- The first wave of person-to-person influence is informational
  - Adoption of hybrid seed corn among farmers in Iowa [Ryan and Gross]
    - Most farmers first learned about hybrid seed corn from salesmen (first wave), but they were later convinced to try it based on the experience of their neighbors (second wave)
  - Adoption of tetracycline by physicians in the US [Coleman, Katz, and Menzel]
    - Doctors learned about tetracycline from publications (first wave), but social connections among doctors were important in their prescribing it (second wave)
  - In both cases, the novelty made it risky to adopt the innovation, but it was ultimately highly beneficial; early adopters tended to have higher socioeconomic status and a tendency to travel; decisions to adopt were made in a social structure

Diffusion of innovations (2)

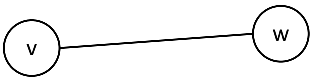
- The second wave of person-to-person influence is direct-benefit
  - Some other examples are the telephone, fax machine, and e-mail, whose adoption depended on communicating with people who already have the new technology
- Common principles that apply to diffusion of innovations [Everett Rogers]
  - Relative advantage: the innovation is an improvement compared to existing practices
  - Simplicity: it must be simple enough for people to implement (not too complex)
  - Observability: people should be aware of other people using the innovation
  - Trialability: people should be able to adopt it gradually
  - Compatibility: it should be compatible with the existing social system
- Homophily can act as a barrier to diffusion (homogeneous communities)
  - Innovations tend to arrive from “outside” the system

A Model for Diffusion in Networks
Part 1: Cascading

Defining a formal model for diffusion

- A person's decisions are based on the decisions of their neighbors
- We define a formal model based on the network graph
  - We could start either from informational effects or direct-benefit effects: an individual knows what his neighbors know, or an individual gets a benefit by adopting what his neighbors use
  - We will define a model based on direct-benefit effects [Stephen Morris]
- A person has social network neighbors
  - The benefits to you of adopting a new behavior increase as more and more of your neighbors adopt it
  - Simple self-interest dictates that you should adopt the new behavior as soon as a sufficiently high proportion of your neighbors have done so!

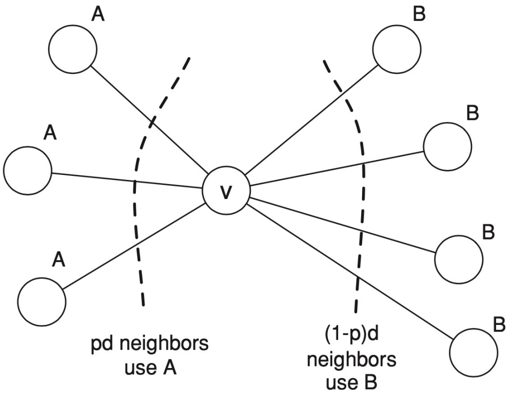
Each edge is a coordination game



		Individual w	
		A	*B*
Individual v	*A*	a, a	0, 0
	B	0, 0	b, b

- We start by looking at two individuals v and w linked by an edge
  - Each individual has a choice between two behaviors, A and B
  - The edge gives them an incentive to have matching behaviors
- We define a simple coordination game (see lecture 3 on game theory)
  - Individuals v and w are the players
  - Behaviors A and B are the strategies
- This is only one edge of the graph, let's look at more edges!

Looking at all neighbors



- Some of v's neighbors adopt A and some adopt B
  - How can v maximize its payoff?
- Assume v has d neighbors and fraction p have behavior A and fraction (1–p) have behavior B
  - If v chooses A, payoff is pda
  - If v chooses B, payoff is (1–p)db
- A is better choice if:

$$p \geq \frac{b}{a+b}$$

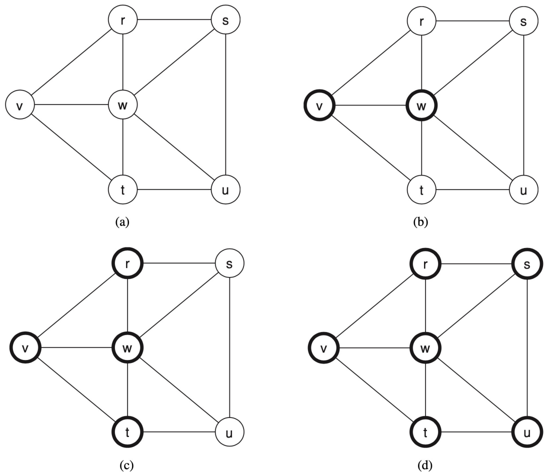
Calculate this as exercise!
What is the intuition?

- We say q=b/(a+b) is the *threshold*

Cascading model for diffusion in networks

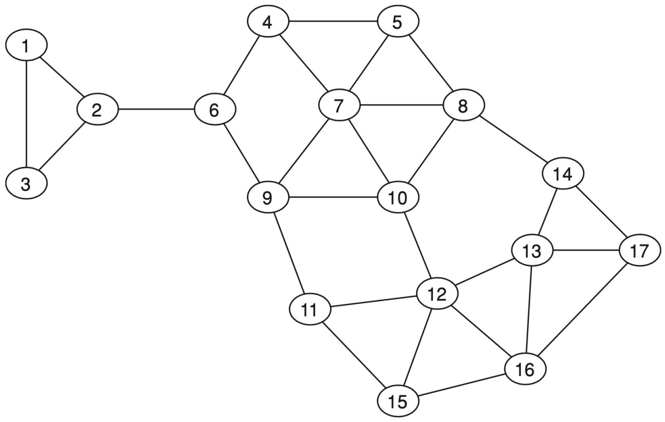
- All networks have two obvious equilibria: all nodes adopt A, or all nodes adopt B
  - Let's see how we can "tip" the network from one equilibrium to the other
  - Are there "intermediate" equilibria, where some parts adopt A and some adopt B?
- Definition of the cascading model
  - All nodes initially use B as their behavior
  - A small set of initial adopters decide to use A
  - The remaining nodes evaluate their payoffs according to the coordination game
  - Some neighbors might decide to switch to A, and so forth, in a cascading fashion
- Let us study this model

Example cascade behavior



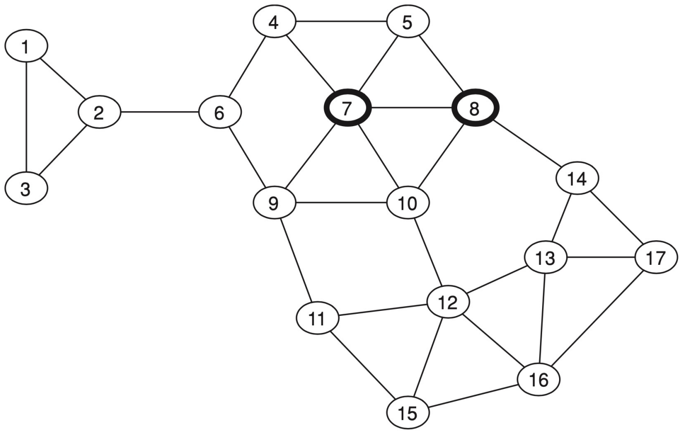
- Suppose the coordination game has a=3 and b=2 (A payoff is 1.5 times B payoff)
  - The threshold value $q = b/(a+b)$ = 2/5
- Nodes v and w are the initial adopters
  - After one step, r and t will adopt A because 2/3 of their neighbors use A, and 2/3 > 2/5
  - After two steps, s and u will adopt A because 2/3 of their neighbors use A, and 2/3 > 2/5
  - It's a chain reaction: nodes converting will allow more nodes to convert, and so on

Second cascade example



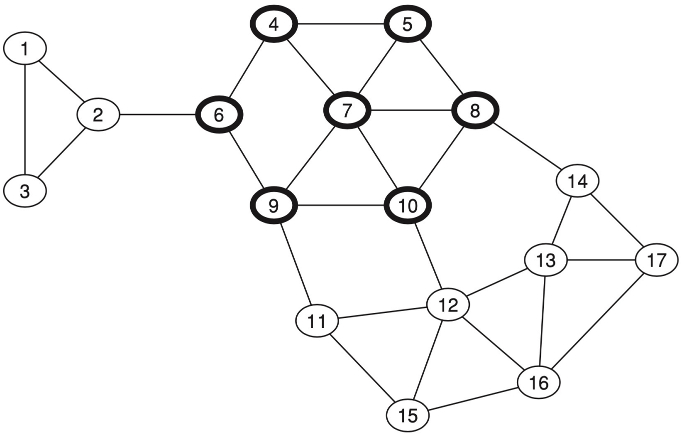
- This is an example where the adoption of A continues for a while but stops before converting all nodes
- Assume again in this example that a=3 and b=2, so the threshold q=2/5
- The initial adopters are nodes 7 and 8

Cascading in the second example



- Nodes 7 and 8 are the initial adopters
- In the next three steps we have:
  - Nodes 5 and 10 switch
  - Nodes 4 and 9 switch
  - Node 6 switches
- Then the cascading stops

Final situation in the second example



- This is the final outcome
- No more nodes will flip
- We call this a cascade of adoptions of A

Complete cascade at threshold q

- We would like to distinguish two possibilities
  i. The cascade runs for a while but then stops without flipping all nodes
  ii. There is a complete cascade, where all nodes flip

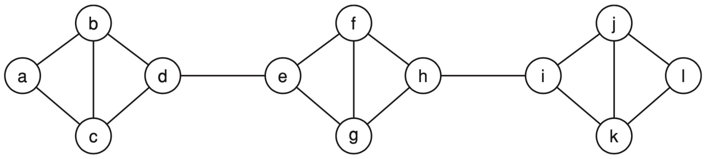
- We introduce the following terminology
  - Consider a set of initial adopters with behavior A and all remaining nodes have behavior B. Nodes repeatedly evaluate the decision whether to switch from B to A using a threshold of q. If the resulting cascade of adoptions of A eventually causes all nodes to switch from B to A, then we say that the set of initial adopters causes a complete cascade at threshold q.

“Viral marketing”

- We make some general observations from the previous example
- Tightly-knit communities can hinder the spread of an innovation
  - A was never able to “leap across the shores” from 8-10 to 11-14
  - There is coexistence between A and B, just like in the real world with political viewpoints or technology use (e.g., Macintosh versus PC)
- There are techniques to push adoption beyond the barriers
  - The maker of A could raise its quality from a=3 to a=4, then q drops from 2/5 to 1/3
    - All nodes will now switch!
  - Another way is to convince a small number of key people to switch to A, for example 12 or 13
    - All nodes will switch!
    - But it is not useful to convince 11 or 14, since nobody else will switch!
- Contrast this with a population-level model (like the direct-benefit model)
  - In a population-level model, it can be very hard for new technology to start! In a network model, it can be much easier, since you only need to worry about your neighbors.

A Model for Diffusion in Networks
Part 2: Clusters

Defining a cluster



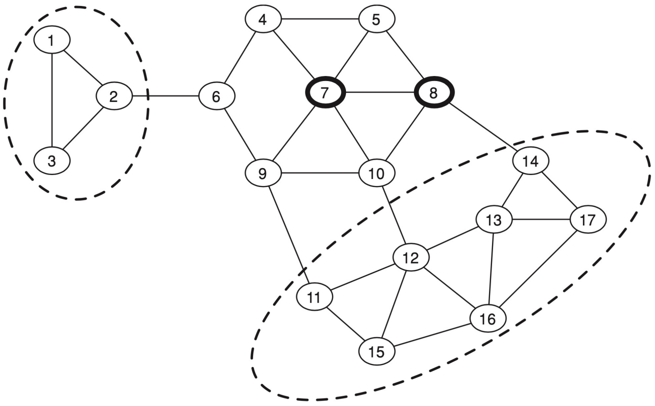
a, b, c, d is a cluster of density 2/3
e, f, g, h is a cluster of density 2/3
i, j, k, l is a cluster of density 2/3

- We saw that a diffusion can stop when it tries to break into a tightly-knit, or dense, community
  - Can we make this intuition precise?
- A first step is to define "densely connected community":
  - Definition: A cluster of density p is a set of nodes such that each node in the set has at least a fraction p of its neighbors in the set

Some observations

- Let's examine this definition of a cluster
- Each node in a cluster has a prescribed fraction of its friends in the cluster, so there is some kind of "cohesion"
- On the other hand, our definition does not imply that nodes have much in common!
  - In any network, the set of all nodes is always a cluster of density 1
  - The union of two clusters of density p is also a cluster of density p *(exercise: explain why!)*
- Clusters can exist simultaneously at many different scales

Relating clusters and cascades

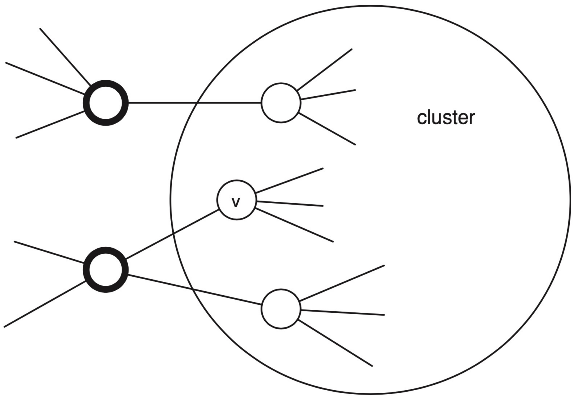


- In the network from our previous example there are two clusters each of density 2/3
  - Density 2/3 > threshold 2/5
  - These correspond precisely to the parts A was unable to break into
  - Could this be a general principle?
- Yes, it is a general principle in our model
  - Clusters are the natural obstacles to cascades!

Cluster-cascade theorem

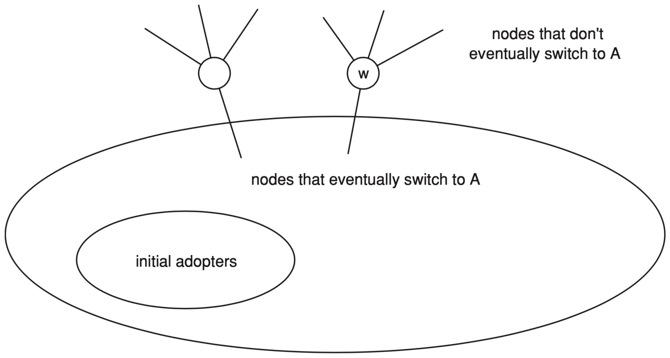
- We formulate the following theorem
- Theorem: Consider a set of initial adopters of behavior A with a threshold q for nodes in the remaining network to adopt behavior A.
  Then the following holds:
  i. If the remaining network contains a cluster of density greater than (1–q) then the set of initial adopters will not cause a complete cascade
  ii. Whenever a set of initial adopters does not cause a complete cascade with threshold q, then the remaining network must contain a cluster of density greater than (1–q)
- This gives a precise characterization of success or failure of a cascade!
  - Basically, a complete cascade is blocked if and only if the remaining network contains a cluster of density greater than (1–q)

Proof of (i): clusters are obstacles to cascades



- Consider an arbitrary network in which behavior A is spreading with threshold q, from a set of initial adopters
  - Suppose the remaining network has a cluster of density > (1–q)
  - We show that no node in the cluster will adopt A
- Assume the opposite, v has adopted A at earliest time step t
  - At that time, the only neighbors of v using A were outside of the cluster
  - But since the cluster has density > (1–q), therefore less than fraction q of v’s neighbors are outside the cluster

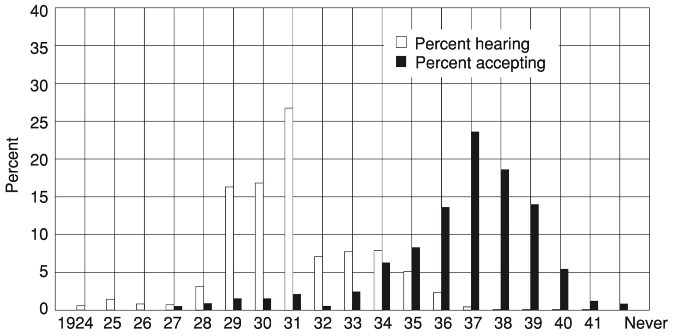
Proof of (ii): clusters are the only obstacles



- Assume a set of initial adopters fails to cause a complete cascade
  - The cascade stops because there are still nodes using B that do not want to switch
- Let S denote the set of nodes using B at the end that do not switch
  - Consider any node w in S
  - Since w does not want to switch, therefore the fraction of its neighbors using A is less than q
  - Therefore the fraction using B is > (1–q)
  - Therefore S is a cluster of density > (1–q)

The Role of Weak Ties
in Diffusion

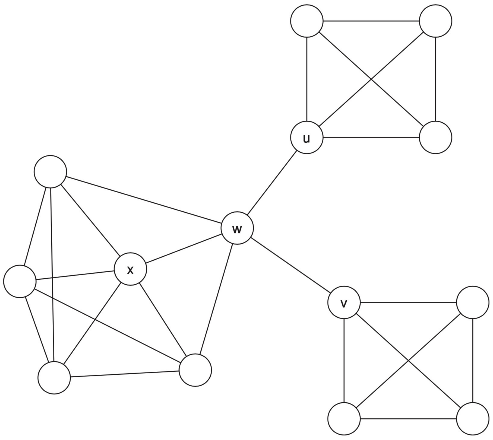
Awareness versus adoption



First wave → Second wave

- There is a crucial difference between learning about a new idea (first wave) and actually adopting it (second wave)
- This figure shows the years of first awareness and first adoption for hybrid seed corn [Ryan and Gross]
- Our cascade model also shows this
  - Nodes are aware of a behavior when a neighbor adopts it, but they only adopt it later (or maybe never)

Weak ties convey awareness but not adoption



- We make the connection with strong and weak ties (remember Chapter 3!)
- Remember that weak ties (acquaintances) often form local bridges
  - They provide access to new sources of information in the network
  - The u-w and w-v links provide access across tightly-knit communities
- But if w and x are initial adopters, then u and v will not adopt!
  - Bridges and local bridges are double-edged swords
  - They convey awareness but they do not encourage adoption
- Weak ties only convey awareness; strong ties are needed for adoption

Application to social movements

- Why do social movements often build support slowly and locally?
- A world-spanning network of weak ties spreads awareness of an idea or a video very quickly (first wave)
  - But actual change in behavior (second wave), namely political mobilization, is much slower and needs to gain momentum in neighborhoods and small communities, through strong ties
- Social movements tend to be risky undertakings
  - Threshold for participating is high, so local bridges are less useful
  - Social movements often spread geographically, because of strong ties
  - Strong ties are more important for recruitment than weak ties

Cascading with Heterogeneous Thresholds

Heterogeneous thresholds

- It’s possible to extend the basic cascade model, keeping its intuitions while adding new intuitions
- We do one extension, namely heterogeneous thresholds
  - Each node values behaviors A and B differently
  - Almost all the analysis we did before will carry over with minor changes
- Heterogeneous thresholds change how cascades happen
  - Influenceable people (with low thresholds) will now favor cascades
  - However, clusters will still block cascades, with minor change in definition

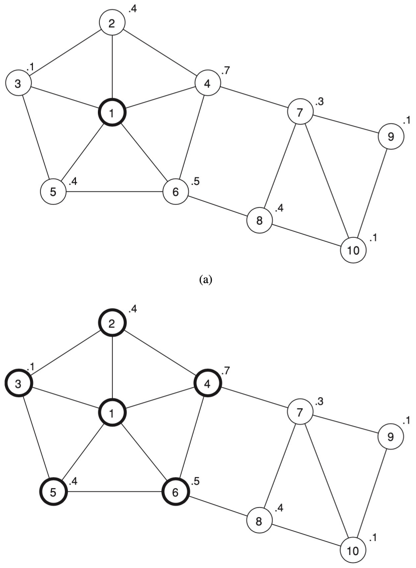
Heterogeneous threshold game

Individual w

		A	*B*
Individual v	*A*	a_v , a_w	0, 0
	B	0, 0	b_v , b_w

- Suppose that each person values behaviors A and B differently
  - For each node v, we have payoff a_v for A and payoff b_v for B
  - This gives a new coordination game
- Previous results still hold, but each node v has its own threshold q_v
  - A is a better choice if: $p \geq \dfrac{b_v}{a_v + b_v} = q_v$

Example with heterogeneous thresholds



- Despite node 1's central position, it would not have converted anybody if it were not for 3's extremely low threshold
  - We need to take into account not just influential nodes but also influenceable people
- Can we also extend the concept of cluster to the heterogeneous case?
  - Yes! A blocking cluster is a set of nodes for which each node v has more than $1 - q_v$ fraction of its friends in the set
  - We can prove a similar theorem as before: a set of initial adopters will cause a complete cascade if and only if the remaining network does not have a blocking cluster

A Model for Collective Action

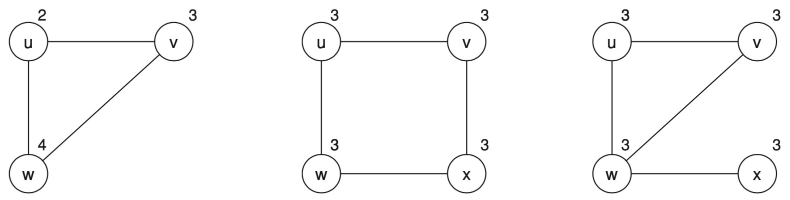
Collective action

- Assume we are organizing a protest under a repressive regime
    - A public demonstration is planned for tomorrow
    - If many people show up, then the government will be weakened
    - If only a few show up, then they will all be arrested
    - What should you do?
- This is a collective action problem
    - An activity provides benefits only if enough people participate
    - It is similar to network effects, but with much less communication
    - Your decision is made difficult by the lack of knowledge of other people’s decisions
- Limiting communication increases the difficulty of collective action
    - Repressive governments limit communication because this weakens the opposition

Pluralistic ignorance

- Pluralistic ignorance is when people have strongly erroneous estimates of how common certain opinions are in the population
  - A majority wants a revolution, but they think that they are a minority
  - A survey in the US in the 1970s showed that a majority of white Americans were against racial segregation, but they believed that they were a minority in their region
- Repressive governments work to increase pluralistic ignorance
  - Most people are against the government but they believe that they are in a small minority, so they take no action
  - An unpopular government can survive for a long time even if there is very strong opposition, enough to topple the government if estimated accurately
- Limiting communication therefore has two related effects:
  1. It increases pluralistic ignorance
  2. It makes collective action difficult

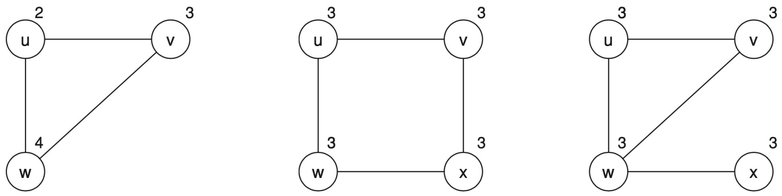
A model for collective action



- An uprising will not occur in the first two networks, but only in the third
- Why is this?

- We define a model for collective action based on personal threshold:
  - Each person in a social network knows about an upcoming action
  - Each person has a personal threshold k that encodes his willingness to participate: “I will join if am sure that at least k people (including myself) will join”
  - Links in the social network encode strong ties, so that each person knows his neighbor’s threshold and neighbors, but nothing else (you only trust people you meet)
- Given a network with a set of thresholds, how do we reason?

Different levels of reasoning



- In the first case, w will certainly never join, therefore v will never join, therefore u will never join (chain of reasoning with three steps)
- In the second case, consider u's view: he knows that v and w each have threshold 3, but he also knows that v and w don't know each other's thresholds
  - Is it safe for u to join? No, because if x's threshold is high (like 5), then v would not join.
  - This is symmetric, so all nodes decide not to join
- In the third case, each of u, v, and w knows that there are three nodes with threshold 3, and they know that they know, and they know that they know that they know, indefinitely. It is common knowledge.
  - Therefore, they know that they can participate and that the others know that too

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Strong ties are needed for collective action

- The nodes' decisions depends on subtle reasoning about what the other nodes know
  - It is highly dependent on the existence of the right strong ties
- For just passing information, a few weak ties are sufficient
  - Loosely coupled communities are sufficient for disseminating information
- For collective action, **strong ties are necessary**
  - Given a tightly coupled community, i.e., many strong ties between members
  - With strong ties the community can can obtain common knowledge. For example, if there is a secret rally where everybody sees everybody.
  - With common knowledge the community can decide on a collective action

Many strong ties ⟹ common knowledge ⟹ collective action

Importance of common knowledge

- Common knowledge is extremely important in open societies
- Many social institutions help people achieve common knowledge
  - A widely publicized speech, an article in a high-circulation newspaper, do not just transmit a message, but they make the listeners aware that many others have also gotten the message
  - This is why freedom of the press and freedom of assembly are important
- Nonpolitical institutions are also important for common knowledge
  - Other institutions, not political, can also play this role, like Super Bowl commercials (in the US), for example the Apple Macintosh commercial during the 1984 Super Bowl
  - After the 2003 US invasion of Iraq, the difference in organization between Sunni and Shiite religious institutions: Friday sermons at Shiite mosques could be centrally coordinated, thus creating common knowledge, while the Sunni establishment lacked comparable structures

Summary of Cascading

Cascading in networks

- Adoption of innovations depends on your neighbors adopting them!
  - It works in two waves: first, information diffusion, and second, adoption
- Idealized model of network cascades and cluster-cascade theorem
  - We defined a model based on direct-benefit effects
  - A cluster of density p: Each node has at least fraction p of neighbors in the set
  - Given a network with a threshold of q=b/(a+b) to adopt a new behavior
  - Cluster-cascade theorem: A complete cascade will occur if and only if the remaining network does not have a cluster of density greater than (1–q)
- Idealized model of collective action
  - Pluralistic ignorance occurs when there is no information diffusion
    - Communication is strongly suppressed, since a few weak ties suffice for information diffusion!
  - For collective action, information diffusion is needed but it is not enough. In addition, it needs many strong ties and preferably enough of them to achieve common knowledge (where everybody knows what everybody else knows)

Course Summary

Course summary

- We have now finished this course on how networks and people influence each other
- The course is based on the book "Networks, Crowds, and Markets".
  This book aims to create a new scientific discipline by combining ideas from many old disciplines:
  - Foundations: graph theory, game theory, and sociology
  - Economics: auctions, markets, intermediaries, negotiation
  - Information networks: Web structure, search, and importance
  - Network dynamics: cascades, "network effects", clusters
  - Network properties: Power laws and the long tail
- We defined many formal models to help us understand real-world phenomena
  - The formal models were idealized (so we can analyze them) and designed to give good intuitions
  - The next slide summarizes these formal models; there are very many of them!
- Where do we go from here?
  - Society and the Internet are continuing their long reciprocal transformation, and research into how people and networks influence each other is still at an early stage. Maybe some day we will have more general laws and not just a collection of partly connected results!
- Advice on how to study for the exam
  - A series of questions on both theory and practice, based on the formal models and results of the course

The Convergence Toolkit
for Understanding the Internet

This chart gives a (non-exhaustive!) list of convergence effects that occur whenever people interact through networks. Real-world networks are *always converging to a changing equilibrium*. Understanding this is important for organizations using networks.

This chart was compiled by Peter Van Roy from the excellent book "Networks, Crowds, and Markets" by David Easley and Jon Kleinberg. The chart is simplified for brevity. Please refer to the book for full explanations.

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- **Social networks, graphs where nodes are humans (chapter 1-5)**
  - Closure: a social-affiliation network (which has both human nodes and center of interest nodes) tends to add links ("friends of friends become friends")
  - Structural balance: a friend/enemy network evolves toward balance:
    2 enemy groups (if strongly balanced) or n enemy groups (if weakly balanced)
- **Games, where players make decisions to get payoff (chapter 6, 8)**
  - Games tend to converge somewhere between two extremes:
    * Nash equilibrium, which models a pure free market
    (which is based on reciprocal best response and indifference principle), and
    * Pareto optimality, which models government regulation
- **Auctions, where a group of players compete for an item (chapter 9)**
  - Traditional ascending-bid (like eBay) are second-price auctions, which tend to converge toward bidding your true value (which is a dominant strategy)
- **Markets, where a group of buyers and sellers interact (chapter 10)**
  - Market prices tend to converge toward market clearing ("supply equals demand"), and prices are a decentralized conflict resolution mechanism
- **Trading networks, adds traders as intermediate nodes (chapter 11)**
  - Economic networks with traders converge in two ways: monopoly maximizes profits to traders and competition maximizes profits to sellers and buyers
  - Increasing the network connectivity increases the overall benefit to society and reduces individual trader profits

- **Negotiation on networks, when nodes bargain (chapter 12)**
  - Negotiation power of a node depends on its position in the network; and negotiation converges according to the following three rules:
  - Stability: convergence tends to remove instabilities
    (instability = ability for a neighbor to sabotage an agreement)
  - Balance: convergence tends to a Nash bargaining solution for all nodes (which is a solution that is considered fair by its participants)
  - Avoiding extremes: humans avoid "all-or-nothing" division of benefits
- **PageRank algorithm, to determine quality of a Web page (chapter 13, 14)**
  - Page importance is modeled as fluid flow on the Web graph; always converging to equilibrium of flow and evaporation (equivalent to random walk with jumps)
- **Cascades, when each decision is based on others (chapter 16, 17, 19)**
  - Information cascades form easily: they form when decisions made sequentially tilt too strongly one way or another, which locks in all future decisions; on the other hand they are easily broken by bringing in new information
  - Direct-benefit cascades ("network effects") get benefits from a large community:
    * Tipping point (unstable equilibrium): below, fraction of users converges to zero, above, it converges to a high value. The goal is to get beyond it.
    * Lowering price: lowering the price can sometimes convert a tipping point into a narrow passageway that converges to a high value
    * Cluster: a tightly-knit community in a network can block network cascades; this can sometimes be avoided by convincing influenceable targets to switch
- **Power law of popularity, when network growth is "organic" (chapter 18)**
  - Preferential attachment: copying earlier decisions (such as links on older Web pages) converges to a power law and gives a significant long tail